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TOPICAL REVIEW

Nonlinear photonic crystals: III. Cubic nonlinearity

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Abstract

Weakly nonlinear interactions between wavepackets in a lossless periodic dielectric medium are studied based on the classical Maxwell equations with a cubic nonlinearity. We consider nonlinear processes such that: (i) the amplitude of the wave component due to the nonlinearity does not exceed the amplitude of its linear component; (ii) the spatial range of a probing wavepacket is much smaller than the dimension of the medium sample, and it is not too small compared with the dimension of the primitive cell. These nonlinear processes are naturally described in terms of the cubic interaction phase function based on the dispersion relations of the underlying linear periodic medium. It turns out that only a few quadruplets of modes have significant nonlinear interactions. They are singled out by a system of selection rules including the group velocity, frequency and phase matching conditions. It turns out that the intrinsic symmetries of the cubic interaction phase stemming from assumed inversion symmetry of the dispersion relations play a significant role in the cubic nonlinear interactions. We also study canonical forms of the cubic interaction phase leading to a complete quantitative classification of all possible significant cubic interactions. The classification is ultimately based on a universal system of indices reflecting the intensity of nonlinear interactions.

1. Introduction

The effect of the spatial periodicity on nonlinear optical processes has been the subject of intensive studies in the physical literature; see [18] for a recent review of the theory of nonlinear photonic crystals, see also [1, 6, 7, 15] and references therein.

In this paper we consider photonic crystals (lossless dielectric periodic media) with cubic nonlinearities based upon the approach developed in our preceding papers [6–9]. We study weakly nonlinear phenomena satisfying the following basic conditions:

(i) the amplitude of the wave component due to the nonlinearity does not exceed the amplitude of its linear component and
(ii) the wavepacket spatial range is much smaller than the dimension of the medium sample.

These phenomena can be naturally studied based on the underlying linear medium as a frame of reference. Our study of the weakly nonlinear phenomena does not require small nonlinear susceptibilities which can be whatever they happen to be. The term 'weak' rather refers to appropriately small initial amplitudes of the electromagnetic (EM) wave. We consider general periodic media in space dimensions $d = 1, 2$ and 3, we do not impose conditions on the structure of the susceptibility tensors.

As in [6, 7] we assume that the EM wave propagation is described by the classical Maxwell equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{1}{c}\partial_t \boldsymbol{B}(\boldsymbol{r}, t) - \frac{4\pi}{c}\boldsymbol{J}_{\mathrm{M}}(\boldsymbol{r}, t), \qquad \nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0, \tag{1}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \frac{1}{c}\partial_t \boldsymbol{D}(\boldsymbol{r}, t) + \frac{4\pi}{c}\boldsymbol{J}_{\mathrm{E}}(\boldsymbol{r}, t), \qquad \nabla \cdot \boldsymbol{D}(\boldsymbol{r}, t) = 0, \tag{2}$$

where $\boldsymbol{H}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ are respectively, the magnetic and electric fields, the magnetic and electric inductions, and $\boldsymbol{J}_{\mathrm{E}}$ and $\boldsymbol{J}_{\mathrm{M}}$ are impressed electric and, so-called, impressed magnetic currents (current sources). It is also assumed that *there are no free electric and magnetic charges*, and, consequently, the fields $\boldsymbol{B}$ and $\boldsymbol{D}$ are divergence free as indicated in equations (1) and (2). Equations (1) and (2) readily imply that the impressed electric and magnetic currents are also divergence free, i.e.

$$\nabla \cdot \boldsymbol{J}_{\mathrm{E}}(\boldsymbol{r}, t) = 0, \qquad \nabla \cdot \boldsymbol{J}_{\mathrm{M}}(\boldsymbol{r}, t) = 0. \tag{3}$$

We use the impressed currents primarily to generate wavepackets playing the key role in the analysis of nonlinear phenomena. For simplicity we consider non-magnetic media, i.e.

$$\boldsymbol{B}(\boldsymbol{r}, t) = \mu \boldsymbol{H}(\boldsymbol{r}, t), \qquad \mu = 1. \tag{4}$$

The material relations between $\boldsymbol{D}$ and $\boldsymbol{E}$ are assumed to be of the standard form, [12]

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P}(\boldsymbol{r}, t; \boldsymbol{E}) \tag{5}$$

where the polarization $\boldsymbol{P}$ includes both the linear and the nonlinear parts

$$\boldsymbol{P}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot)) = \boldsymbol{P}^{(1)}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot)) + \boldsymbol{P}_{\mathrm{NL}}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot)). \tag{6}$$

To quantify the relative impact of the nonlinearity we introduce a dimensionless constant α_0 and scale all the fields as follows:

$$\boldsymbol{J}_{\mathrm{E}} \to \alpha_0 \boldsymbol{J}_{\mathrm{E}}, \qquad \boldsymbol{J}_{\mathrm{M}} \to \alpha_0 \boldsymbol{J}_{\mathrm{M}}, \qquad \boldsymbol{E} \to \alpha_0 \boldsymbol{E}, \tag{7}$$

$$\boldsymbol{D} \to \alpha_0 \boldsymbol{D}, \qquad \boldsymbol{H} \to \alpha_0 \boldsymbol{H}, \qquad \boldsymbol{B} \to \alpha_0 \boldsymbol{B}. \tag{8}$$

Then the magnitude of the rescaled nonlinearity $\tilde{\boldsymbol{P}}_{\mathrm{NL}}(\tilde{\boldsymbol{E}})$ with a cubic leading term is of order $\alpha = \alpha_0^2$ for $\alpha_0 \ll 1$, and the material relation becomes

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi[\boldsymbol{P}^{(1)}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot)) + \alpha \boldsymbol{P}_{\mathrm{NL}}(\boldsymbol{r}, t; \boldsymbol{E}; \alpha)], \qquad \alpha = \alpha_0^2 \ll 1, \tag{9}$$

where α measures the relative magnitude of the nonlinearity. We allow for $\boldsymbol{P}_{\mathrm{NL}}$ a general analytic dependence in $\boldsymbol{E}$ with the leading term being cubic, namely

$$\boldsymbol{P}_{\mathrm{NL}}(\boldsymbol{r}, t; \boldsymbol{E}) = \boldsymbol{P}^{(3)}(\boldsymbol{r}, t; \boldsymbol{E}) + \sum_{h>3}^{\infty} \boldsymbol{P}^{(h)}(\boldsymbol{r}, t; \boldsymbol{E}; \alpha), \tag{10}$$

where $\boldsymbol{P}^{(h)}(\boldsymbol{E})$ are h-linear (tensorial) operators. Observe, that, in view of (9), the leading term $\boldsymbol{P}^{(3)}(\boldsymbol{E})$ does not depend on α. For a cubic nonlinearity the nonlinear part $\boldsymbol{P}_{\mathrm{NL}}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot))$ is often assumed to be homogeneous in $\boldsymbol{E}(\cdot)$ of the order 3, that is higher order terms being neglected, [12, 13]. According to classical nonlinear optics, see [12] (section 2), $\boldsymbol{P}^{(h)}$ has the following form:

$$\boldsymbol{P}^{(h)}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot); \alpha) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} P^{(h)}(\boldsymbol{r}; t - t_1, \ldots, t - t_h; \alpha) \vdots \prod_{j=1}^{h} \boldsymbol{E}(\boldsymbol{r}, t_j)\, \mathrm{d}t_j, \tag{11}$$

where $P^{(h)}$ is the so-called h-order polarization response function. For fixed $\boldsymbol{r}$ and $t - t_j$ the quantity $\boldsymbol{P}^{(h)}$ is a h-linear tensor acting on the components of $\boldsymbol{E}(\boldsymbol{r}, t_j)$. Here we do not make assumptions on the structure of this tensor. This form of the polarization response function given in (11) takes explicitly into account *two fundamental properties of the medium: the time invariance and the causality*, [12, section 2].

The linear part $\boldsymbol{P}^{(1)}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot))$ of the total polarization is given by

$$\boldsymbol{P}^{(1)}(\boldsymbol{r}, t; \boldsymbol{E}(\cdot)) = \boldsymbol{\chi}^{(1)}(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}, t), \tag{12}$$

where $\boldsymbol{\chi}^{(1)}(\boldsymbol{r})$ is the tensor of linear susceptibility. For simplicity of rigorous argumentation we assume that $\boldsymbol{\chi}^{(1)}(\boldsymbol{r})$ does not depend on the frequency that, from the physical point of view, efficiently binds us to a certain frequency range. We would like to emphasize that this simplifying assumption does not affect the analysis of nonlinear interactions since it takes as a 'starting point' the dispersion relations of the linear medium, whatever they happen to be [6].

It is preferable to deal with divergence-free fields, [6], and for that reason we choose $\boldsymbol{D}$ to be our basic field. To implement that we recast (9) as

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{\eta}^{(1)}(\boldsymbol{r})\boldsymbol{D}(\boldsymbol{r}, t) - \alpha \boldsymbol{S}(\boldsymbol{r}, t; \boldsymbol{D}; \alpha), \tag{13}$$

$$\boldsymbol{\eta}^{(1)}(\boldsymbol{r}) = [\varepsilon^{(1)}(\boldsymbol{r})]^{-1}, \qquad \varepsilon^{(1)}(\boldsymbol{r}) = 1 + 4\pi\boldsymbol{\chi}^{(1)}(\boldsymbol{r}), \tag{14}$$

where $\varepsilon^{(1)}(\boldsymbol{r})$ and $\boldsymbol{\eta}^{(1)}(\boldsymbol{r})$ are respectively, tensors of the dielectric permittivity and the impermeability. The latter is commonly used in studies of electro-optical effects (Pockels and Kerr effects), [20, section 7], [17, sections 6.3, 18.1].

The dielectric properties of the periodic medium are assumed to vary periodically in space. In other words, the tensors $\boldsymbol{\chi}^{(1)}(\boldsymbol{r})$, $\boldsymbol{\eta}^{(1)}(\boldsymbol{r})$ and $\boldsymbol{P}_{\mathrm{NL}}(\boldsymbol{r}, t; \boldsymbol{E}; \alpha)$, $\boldsymbol{S}(\boldsymbol{r}, t; \boldsymbol{D}; \alpha)$ are periodic functions of the position $\boldsymbol{r}$. In particular, if the lattice of periods is cubic with lattice constant L_0, and $\mathbb{Z}^3$ is the lattice of integer valued vectors $\boldsymbol{n}$, then the following periodicity conditions hold for every $\boldsymbol{n}$ from $\boldsymbol{Z}^3$:

$$\boldsymbol{\eta}^{(1)}(\boldsymbol{r} + L_0\boldsymbol{n}) = \boldsymbol{\eta}^{(1)}(\boldsymbol{r}), \qquad P^{(h)}(\boldsymbol{r} + L_0\boldsymbol{n}; t_1, \ldots, t_h; \alpha) = P^{(h)}(\boldsymbol{r}; t_1, \ldots, t_h; \alpha). \tag{15}$$

The case of a non-cubic lattice with different periods L_j in different directions can be considered similarly; we restrict ourselves to the cubic case for simplicity. Substituting $\boldsymbol{E}$ determined by (13) into the Maxwell equations (1), (2), we rewrite them in the following concise form:

$$\partial_t \boldsymbol{U} = -\mathrm{i}\mathcal{M}\boldsymbol{U} + \alpha \boldsymbol{F}_{\mathrm{NL}}(\boldsymbol{U}) - \boldsymbol{J}; \qquad \boldsymbol{U}(t) = 0 \qquad \text{for } t \leqslant 0, \tag{16}$$

where

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{bmatrix}, \qquad \mathcal{M}\boldsymbol{U} = \mathrm{i}\begin{bmatrix} \nabla \times \boldsymbol{B} \\ -\nabla \times (\boldsymbol{\eta}^{(1)}(\boldsymbol{r})\boldsymbol{D}) \end{bmatrix}, \tag{17}$$

$$\boldsymbol{J} = 4\pi \begin{bmatrix} \boldsymbol{J}_{\mathrm{E}} \\ \boldsymbol{J}_{\mathrm{M}} \end{bmatrix}, \qquad \boldsymbol{F}_{\mathrm{NL}}(\boldsymbol{U}) = \begin{bmatrix} \boldsymbol{0} \\ \nabla \times \boldsymbol{S}(\boldsymbol{r}, t; \boldsymbol{D}) \end{bmatrix}, \tag{18}$$

assuming everywhere that all the fields $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{J}_{\mathrm{E}}$ and $\boldsymbol{J}_{\mathrm{M}}$ are divergence free. We also assume that the medium is at rest for all negative times by requiring the impressed currents $\boldsymbol{J}$ vanish for all negative times, i.e.

$$\boldsymbol{J}(t) = 0 \qquad \text{for } t \leqslant 0. \tag{19}$$

As it is shown in [6, 7], a number of nonlinear phenomena of interest are described in terms of the dispersion relations of the underlying linear medium represented by the linear Maxwell operator $\mathcal{M}$ with periodic coefficients. The spectral properties of $\mathcal{M}$, in turn, are based on the Floquet–Bloch theory, [5, 16]. Namely, we remind that all the eigenvalues and the eigenmodes of $\mathcal{M}$ are parametrized by two indices: zone (band) number $n = 1, 2, \ldots$, and the quasimomentum $\boldsymbol{k}$ from the so-called *Brillouin zone*. The Brillouin zone in our case is the cube $[-\pi, \pi]^d$. We assume that the positive eigenvalues $\omega_n(\boldsymbol{k})$ are ordered as follows:

$$0 \leqslant \omega_1(\boldsymbol{k}) \leqslant \omega_2(\boldsymbol{k}) \leqslant \cdots, \qquad \boldsymbol{k} \text{ in } [-\pi, \pi]^d, \tag{20}$$

where $d = 1, 2, 3$ is the space dimension. We call $\boldsymbol{k}$ a simple point of nth dispersion relation $\omega_n(\boldsymbol{k})$ when $\omega_n(\boldsymbol{k}) \neq \omega_{n+1}(\boldsymbol{k})$, $\omega_n(\boldsymbol{k}) \neq \omega_{n-1}(\boldsymbol{k})$. If $\omega_n(\boldsymbol{k}) = \omega_{n+1}(\boldsymbol{k})$ we call $\boldsymbol{k}$ a multiple (or *band-crossing*) point of $\omega_n(\boldsymbol{k})$ and $\omega_{n+1}(\boldsymbol{k})$. Note that the operator $\mathcal{M}$ has the following property: if ω is an eigenfrequency then $-\omega$ is an eigenfrequency as well. To take into account the negative eigenfrequency we introduce the pairs

$$\bar{n} = (\zeta, n) \qquad \text{where } \zeta = \pm 1,\ n = 1, 2, \ldots, \tag{21}$$

and set

$$\omega_{\bar{n}}(\boldsymbol{k}) = \zeta \omega_n(\boldsymbol{k}), \qquad \text{for } \bar{n} = (\zeta, n). \tag{22}$$

The functions $\omega_n(\boldsymbol{k})$ are 2π periodic functions of k_1, k_2, k_3. Recall that for the classical Maxwell equations the corresponding Bloch eigenmodes $\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k})$ satisfy the following relations:

$$\mathcal{M}\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k}) = \omega_{\bar{n}}(\boldsymbol{k})\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k}), \qquad \tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r} + \boldsymbol{m}, \boldsymbol{k}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{m}}\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k}), \qquad \boldsymbol{m} \text{ in } \mathbb{Z}^d, \tag{23}$$

$\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k})$ being divergence free. For every fixed quasimomentum $\boldsymbol{k}$ the eigenfunctions $\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{k})$ with different $\bar{n}$ form an orthonormal basis.

Commonly cubic nonlinearities arise in a central symmetric bulk medium with relevant properties being invariant with respect to the central symmetry reflection S_0: $\boldsymbol{r} \to -\boldsymbol{r}$. Because of fundamental symmetry constraints at the microscopic level most of the dielectric materials possessing cubic nonlinearities are also symmetric with respect to the *inversion transformation* $\boldsymbol{k} \leftrightarrow -\boldsymbol{k}$. So, within this paper we always assume the following *inversion symmetry* condition to hold:

$$\omega_n(\boldsymbol{k}) = \omega_n(-\boldsymbol{k}) \qquad \text{for all } n = 1, 2, \ldots \text{ and } \boldsymbol{k} \in [-\pi, \pi]^d, \tag{24}$$

which is a special case of Wigner time-reversal symmetry; see [14]. Note that this relation implies the following equality for the gradients $\omega_n'(\boldsymbol{k}) = \nabla\omega_n(\boldsymbol{k})$ (the *group velocities*):

$$\omega_n'(\boldsymbol{k}) = -\omega_n'(-\boldsymbol{k}). \tag{25}$$

The inversion symmetry condition (24) plays a very important role in cubic nonlinear interactions (see, for comparison, [10] for the analysis of nonlinear interactions when the inversion symmetry does not apply). In this paper we study the effects related only to the inversion symmetry with the understanding that other symmetries can be treated within the same framework as it is discussed in section 3.4. It is assumed that after taking all the symmetries into account the dielectric medium is a *'generic'* one. A more precise meaning of the term generic is provided in section 2.

Mode interactions analysis is naturally based on the Floquet–Bloch modal expansions for every field $\boldsymbol{U}(\boldsymbol{r})$ of interest, i.e.

$$\boldsymbol{U}(\boldsymbol{r}) = \frac{1}{(2\pi)^d} \sum_{\bar{n}} \int_{[-\pi,\pi]^d} \tilde{U}_{\bar{n}}(\boldsymbol{k})\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k})\, \mathrm{d}\boldsymbol{k}, \tag{26}$$

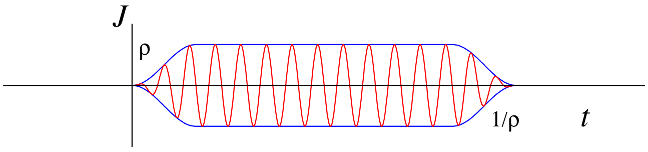

Figure 1. Impressed current J in the form of the wavepacket of the amplitude of order ϱ and of time length of order $1/\varrho$ causes the medium nonlinear response. Based on this response we estimate the rates of energy exchange between different modes.

(This figure is in colour only in the electronic version)

with $\tilde{U}_{\bar{n}}(\boldsymbol{k})$ being the scalar Bloch amplitudes of $\boldsymbol{U}(\boldsymbol{r})$, and $\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{k})$ being eigenmodes (generalized eigenfunctions) of the linear Maxwell operator $\mathcal{M}$. As we have shown in [6] and [7], the substantial part of the nonlinear mode interaction analysis can be carried out based upon the dispersion relations $\omega_n(\boldsymbol{k})$ of the interacting modes and the very general smoothness properties of the eigenmodes $\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{k})$.

2. Cubic nonlinear interactions

Though the cubic nonlinear interactions are described essentially by the same formalism as quadratic ones, [6, 7], there are some important differences extending beyond the evident difference of the number of interacting modes, which are respectively four and three for cubic and quadratic nonlinearities. The qualitative difference stems from the inversion symmetry condition (24) implying consequently related symmetries of the cubic interactions phase. In contrast, the inversion symmetry condition (24) does not affect the quadratic interactions to such a degree. It also turns out, as a consequence of the mentioned symmetries of the cubic interaction phase, that one can explicitly identify quadruplets of modes satisfying the group velocity matching (GVM) condition which is crucial for stronger mode interactions, [7].

In the following sections we focus on those details of the formalism of [6, 7] that are special for cubic nonlinearities.

2.1. First nonlinear response and interacting quadruplets of modes

For the clarity of the argument we assume the photonic crystal occupies the entire space. As in [6, 7] we probe the dielectric medium with the excitation current $\boldsymbol{J}$ of sufficiently small amplitude α_0 and of the relative bandwidth $\varrho \sim \Delta\omega/\omega_0$, where $\Delta\omega$ is the frequency bandwidth of the wavepacket and ω_0 is its carrier frequency (see figure 1).

We choose the excitation currents $\boldsymbol{J}$ to be wavepackets with modal coefficients of the form

$$\boldsymbol{J}_{\bar{n}}(\boldsymbol{k}, t) = \varrho \mathrm{e}_{\bar{n}}^{-\mathrm{i}\omega_{\bar{n}}(\boldsymbol{k})t} j_{\bar{n}}(\boldsymbol{k}, \tau)\tilde{\boldsymbol{G}}_{\bar{n}}(\boldsymbol{k}), \qquad j_{\bar{n}}(\boldsymbol{k}, \tau) = 0 \qquad \text{for } \tau = \varrho t \leqslant 0, \tag{27}$$

where τ is the so-called 'slow time', $\rho \ll 1$. Note that if $\alpha = 0$ equation (16) evidently becomes linear. If the excitation current $\boldsymbol{J}$ is chosen as an appropriate wavepacket, the solution $\boldsymbol{U}^{(0)}$ to the linear (with $\alpha = 0$) Maxwell equations becomes the zero-order approximation to the solution of the nonlinear Maxwell equation (16), [6]. Its modal components can be represented as

$$\boldsymbol{U}_{\bar{n}}^{(0)}(t) = \tilde{V}_{\bar{n}}^{(0)}(\boldsymbol{k}, \tau)\boldsymbol{G}_{\bar{n}}(\boldsymbol{r}, \boldsymbol{k})\mathrm{e}^{-\mathrm{i}\omega_{\bar{n}}(\boldsymbol{k})t}, \qquad \tau = \varrho t, \tag{28}$$

where $\tilde{V}_{\bar{n}}^{(0)}(\boldsymbol{k}, \tau)$ are slowly varying time amplitudes of the time harmonic carrier waves

$\boldsymbol{G}_{\bar{n}}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\omega_{\bar{n}}(\boldsymbol{k})t}$. The amplitudes are explicitly expressed in terms of $\boldsymbol{J}$:

$$\tilde{V}_{\bar{n}}^{(0)}(\boldsymbol{k},\tau) = -\int_0^\tau j_{\bar{n}}(\boldsymbol{k},\tau_1)\,\mathrm{d}\tau_1. \tag{29}$$

We study and classify cubic mode interactions based on the *first nonlinear response* $\boldsymbol{U}^{(1)}(t)$ defined by

$$\boldsymbol{U}(t) = \boldsymbol{U}^{(0)}(t) + \alpha\boldsymbol{U}^{(1)}(t) + \mathrm{O}(\alpha^2). \tag{30}$$

Its Bloch expansion has the form, [6, 7]

$$\boldsymbol{U}_{\bar{n}}^{(1)}(t) = \tilde{V}_{\bar{n}}^{(1)}(\boldsymbol{k},\tau)\boldsymbol{G}_{\bar{n}}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\omega_{\bar{n}}(\boldsymbol{k})t}, \qquad \tau = \varrho t, \tag{31}$$

$$\tilde{V}_{\bar{n}}^{(1)}(\boldsymbol{k},\tau) = \frac{1}{\varrho}\sum_{\bar{n}',\bar{n}''}\int_0^\tau\int_{[-\pi,\pi]^d}\exp\left\{\mathrm{i}\phi_{\bar{n}}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')\frac{\tau_1}{\varrho}\right\}\breve{Q}_{\bar{n}}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')\tilde{V}_{\bar{n}'}^{(0)}(\boldsymbol{k}',\tau_1)$$
$$\times\,\tilde{V}_{\bar{n}''}^{(0)}(\boldsymbol{k}'',\tau_1)\tilde{V}_{\bar{n}'''}^{(0)}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'',\tau_1)\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\boldsymbol{k}''\,\mathrm{d}\tau_1, \tag{32}$$

with the *mode interaction phase function (interaction phase)*

$$\phi_{\bar{n}}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') = \omega_{\bar{n}}(\boldsymbol{k}) - \omega_{\bar{n}'}(\boldsymbol{k}') - \omega_{\bar{n}''}(\boldsymbol{k}'') - \omega_{\bar{n}'''}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}''), \tag{33}$$

where $\omega_{\bar{n}}(\boldsymbol{k})$ are given by (22), $\breve{Q}_{\bar{n}}$ is a coefficient depending on the indices $\vec{n} = (\bar{n},\bar{n}',\bar{n}'',\bar{n}''')$ and the quasimomenta $\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''$, and $\phi_{\bar{n}}$ is 2π-periodic with respect to the $\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''$ phase function (see [6] for the reduction to this form). Observe that the form of the interaction phase $\phi_{\bar{n}}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')$ indicates that for a cubic nonlinearity the modes interact in quadruplets. Formula (32) suggests that the integral

$$I_{\bar{n}}(\boldsymbol{k},\tau) = \frac{1}{\varrho}\int_0^\tau\int_{[-\pi,\pi]^3}\exp\left\{\phi_{\bar{n}}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')\frac{\tau_1}{\varrho}\right\}A(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'',\tau_1)\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\boldsymbol{k}''\,\mathrm{d}\tau_1, \tag{34}$$

$$A(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'',\tau_1) = \tilde{Q}_{\bar{n}}(\tilde{\boldsymbol{k}})\tilde{V}_{\bar{n}'}^{(0)}(\boldsymbol{k}',\tau_1)\tilde{V}_{\bar{n}''}^{(0)}(\boldsymbol{k}'',\tau_1)\tilde{V}_{\bar{n}'''}^{(0)}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'',\tau_1), \tag{35}$$

represents the nonlinear impact of all triads of modes $(\bar{n}',\boldsymbol{k}')$, $(\bar{n}'',\boldsymbol{k}'')$, $(\bar{n}''',\boldsymbol{k}''')$ onto the mode $(\bar{n},\boldsymbol{k})$. We refer to the integrals (34) as the *nonlinear oscillatory interaction integrals*, whereas the interaction process is often referred to as *four-wave mixing*. We would like to emphasize that since the propagating wavepacket always has a finite spatial range, its spectrum consists of a continuum of modes, and all of them are involved in the four-wave mixing. But as it is shown in [6] for quadratic interactions and will be shown here for cubic interactions, almost all interactions, except for a few stronger ones, are insignificant and vanish faster than any power of ϱ as $\varrho \to 0$. This fact reduces the asymptotic approximation of the integral $I_{\bar{n}}(\boldsymbol{k},\tau)$ to a finite sum of contributions coming from a few stronger interacting quadruplets of modes.

As in the preceding papers we classify the nonlinear interaction based on the rate of decay of the oscillatory interaction integrals (32) as $\varrho \to 0$. *The asymptotic behaviour of* $\tilde{V}_{\bar{n}}^{(1)}(\boldsymbol{k},\tau)$ *in (32) as* $\varrho \to 0$ *is determined primarily by the interaction phase* $\phi_{\bar{n}}$. Its form (33) signifies the well known fact that the nonlinear interactions for a cubic nonlinearity occur through quadruplets of modes $(\bar{n},\boldsymbol{k})$, $(\bar{n}',\boldsymbol{k}')$, $(\bar{n}'',\boldsymbol{k}'')$ and $(\bar{n}''',\boldsymbol{k}''')$ with the corresponding dispersion relations $\omega_{\bar{n}}(\boldsymbol{k})$, $\omega_{\bar{n}'}(\boldsymbol{k}')$, $\omega_{\bar{n}''}(\boldsymbol{k}'')$ and $\omega_{\bar{n}'''}(\boldsymbol{k}''')$. The representation (32), (33) explicitly takes into account *the phase matching condition*

$$\boldsymbol{k}''' = \boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'' \tag{36}$$

for the interacting modes that follow from the medium periodicity. The periodicity of $\omega_{\bar{n}}(\boldsymbol{k})$ implies that this equation is understood modulo $(2\pi\mathbb{Z})^d$; see the remark at the end of the following section for details.

To analyse the interactions, we look at the impact of a triad of modes $(\bar{n}',\boldsymbol{k}')$, $(\bar{n}'',\boldsymbol{k}'')$ and $(\bar{n}''',\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')$ onto a mode $(\bar{n},\boldsymbol{k})$, and observe that the amplitude of the first nonlinear response $\tilde{V}_{\bar{n}}^{(1)}(\boldsymbol{k},\tau)$ depends on the amplitudes $\tilde{V}_{\bar{n}'}^{(0)}(\boldsymbol{k}',\tau_1)$, $\tilde{V}_{\bar{n}''}^{(0)}(\boldsymbol{k}'',\tau_1)$ and $\tilde{V}_{\bar{n}'''}^{(0)}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'',\tau_1)$, $\tau_1\leqslant\tau$, of the linear response. Let us fix $\tau>0$, and denote the contribution of amplitudes $\tilde{V}_{\bar{n}'}^{(0)}(\boldsymbol{k}')$, $\tilde{V}_{\bar{n}''}^{(0)}(\boldsymbol{k}'')$ and $\tilde{V}_{\bar{n}'''}^{(0)}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')$ to the amplitude $\tilde{V}_{\bar{n}}^{(1)}(\boldsymbol{k})$ by $\tilde{V}_{\bar{n}}^{(1)}(\bar{n},\bar{n}',\bar{n}'',\bar{n}''',\boldsymbol{k}'',\boldsymbol{k}',\boldsymbol{k})$. Using the same arguments as given in [6, 7] we show that for $\varrho\to 0$ the first nonlinear response $\tilde{V}_{\bar{n}}^{(1)}$ always vanishes as a power of ϱ, namely

$$\tilde{V}_{\bar{n}}^{(1)}(\bar{n},\bar{n}',\bar{n}'',\bar{n}''',\boldsymbol{k}'',\boldsymbol{k}',\boldsymbol{k})\sim\varrho^{q_0-1},\qquad \varrho\to 0 \text{ where } 0<q_0\leqslant\infty, \tag{37}$$

where the index $q_0=q_0(\bar{n},\bar{n}',\bar{n}'',\bar{n}''',\boldsymbol{k}'',\boldsymbol{k}',\boldsymbol{k})$ describes the intensity of the interaction. If, for instance, the modes $(\bar{n},\boldsymbol{k})$, $(\bar{n}',\boldsymbol{k}')$, $(\bar{n}'',\boldsymbol{k}'')$ and $(\bar{n}''',\boldsymbol{k}''')$ are chosen 'at random' then the above index q_0 is infinite and, consequently, the mode interaction is weaker than any power of ϱ as $\varrho\to 0$.

The strongly interacting modes for which q_0 is finite (in fact, for such modes $q_0\leqslant d$, with d being the space dimension) are determined by the system of selection rules following from the asymptotic analysis of the oscillatory integral (34).

2.2. *The system of selection rules*

To single out and classify significant nonlinear interactions one has to study the asymptotic behaviour of the integral (34) as $\varrho\to 0$. An analysis of the interaction integrals $I_{\bar{n}}(\boldsymbol{k},\tau)$ of the form (34) along the lines of [6] shows that stronger nonlinear interactions can be found with the help of three selection rules. The first one, known as the *phase matching condition* (36), is already built in the very form of the interaction integral $I_{\bar{n}}(\boldsymbol{k},\tau)$ where $\boldsymbol{k}'''=\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}''$. The second and the third selection rules, respectively, the GVM rule and the *frequency matching* (FM) rule, are as follows:

$$\nabla\omega_{\bar{n}'}(\boldsymbol{k}'_*)=\nabla\omega_{\bar{n}''}(\boldsymbol{k}'')=\nabla\omega_{\bar{n}'''}(\boldsymbol{k}'''),\qquad \boldsymbol{k}'''=\boldsymbol{k}-\boldsymbol{k}'_*-\boldsymbol{k}''_*, \tag{38}$$

$$\omega_{\bar{n}}(\boldsymbol{k})-\omega_{\bar{n}'}(\boldsymbol{k}')-\omega_{\bar{n}''}(\boldsymbol{k}'')-\omega_{\bar{n}'''}(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')=0. \tag{39}$$

Observe that the GVM condition (38) is the system of two d-component equations, and the FM condition (39) is a scalar equation. Evidently, for a fixed vector $\boldsymbol{k}$ the system (38), (39) consists of $2d+1$ scalar equations for the total of the $2d$ variables which are components of the vectors $\boldsymbol{k}'$ and $\boldsymbol{k}''$. Consequently the system is formally overdetermined. Though, thanks to the special structure of the system (38), (39) and the inversion symmetry condition (24), *the selection rules system (38), (39) always has solutions.*

As we have already pointed out in [7] the GVM rule is the most important selection rule. Its significance follows from for the fact that if the group velocity condition (38) does not hold, the corresponding mode interaction is very weak, namely the interaction integrals

$$I_{\bar{n}}(\boldsymbol{k})=\frac{1}{\varrho}\int_{[-\pi,\pi]^d}\exp\left\{\mathrm{i}\phi_{\bar{n}}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')\frac{\tau_1}{\varrho}\right\}A(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'',\tau_1)\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\boldsymbol{k}'' \tag{40}$$

included in (34) vanish faster than any power of ϱ as $\varrho\to 0$. In contrast, if (38) holds, the integral $I_{\bar{n}}(\boldsymbol{k})$ in (40) is of order ϱ^{q_0-1} where q_0 is a finite number determined by the properties of the phase function $\phi_{\bar{n}}$ at a relevant point, [6]. It turns out that for most of the stronger interacting quadruplets we have $q_0=d$. It is shown in [6] that when the GVM condition holds and the FM condition (39) does not, the magnitude of the interaction integral $I_{\bar{n}}(\boldsymbol{k},\tau)$ is of order ϱ^{q_0} with a finite q_0, whereas in the case when both the group velocity (38) and the FM (39) conditions hold it is of order ϱ^{q_0-1}. There are special quadruplets with corresponding indices $q_0<d$, these are the most strongly interacting quadruplets. It turns out that the index q_0 takes on only a few universal values, the most important of them are collected in the tables presented in section 2. Note that a smaller interaction index q_0 corresponds to a stronger interaction.

In this paper we consider only the strongest possible cubic interactions satisfying the full system of selection rules (38), (39). The cases when the system (38), (39) is not fully satisfied essentially are considered in [6].

Now let us briefly recall the origin of the system of the selection rules. We want to single out those points $(\boldsymbol{k}'_*, \boldsymbol{k}''_*)$ which provide the most significant contributions to the interaction integral $I_{\vec{n}}(\boldsymbol{k}, \tau)$ as $\varrho \to 0$. As it follows from the *stationary phase method*, for any fixed $\boldsymbol{k}$ and $\vec{n}$, the most significant contributions come from the so-called *critical points* $(\boldsymbol{k}'_*, \boldsymbol{k}''_*)$ solving the following two d-component equations:

$$\nabla_{k'}\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}'_*, \boldsymbol{k}''_*) = \mathbf{0}, \qquad \nabla_{k''}\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}'_*, \boldsymbol{k}''_*) = \mathbf{0}. \tag{41}$$

In view of (33), equations (41) can be recast as (38), that is critical points of the phase are the points for which the GVM rule holds. The FM condition (39) is derived from the analysis of the integral with respect to τ in (34) (see [6]).

2.2.1. Cubic interaction phase function and its symmetries. When applying the selection rules (38), (39) one has to take into account the intrinsic symmetries of the cubic interaction phase function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ as defined by (33) with respect to the transformations

$$\begin{array}{lll} (n', \boldsymbol{k}') \leftrightarrow (n'', \boldsymbol{k}''), & (n', \boldsymbol{k}') \leftrightarrow (n''', \boldsymbol{k}''') & \text{and} \\ \boldsymbol{k}' \leftrightarrow -\boldsymbol{k}', & \boldsymbol{k}'' \leftrightarrow -\boldsymbol{k}'', & \boldsymbol{k}''' \leftrightarrow -\boldsymbol{k}'''. \end{array} \tag{42}$$

We study the properties of the interaction phase $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ under the assumption that all the dispersion relations $\omega_n(\boldsymbol{k})$ are generic in the sense that they do not have any 'hidden symmetries', and that all their degeneracies are *robust*. The concept '*of being generic*' can be described more 'constructively' based on admissible perturbations of $\omega_n(\boldsymbol{k})$ which are assumed to satisfy the following properties.

Admissible perturbations. For any set of functions $\omega_n(\boldsymbol{k})$, $n = 1, 2, \ldots$, satisfying the inversion symmetry relation (24), the corresponding set of perturbations $\delta\omega_n(\boldsymbol{k})$, $n = 1, 2, \ldots$, are called *admissible* if: (i) all $\delta\omega_n(\boldsymbol{k})$ satisfy the inversion symmetry relation (24); (ii) all functions $\delta\omega_n(\boldsymbol{k})$ are infinitely differentiable in $\boldsymbol{k}$.

Note that for admissible perturbations the values of $\delta\omega_{n'}(\boldsymbol{k}')$ and $\delta\omega_{n''}(\boldsymbol{k}'')$ are independent in the vicinity of any two simple (that is, not band-crossing) points $\boldsymbol{k}'_*$, $\boldsymbol{k}''_*$ if $n' \neq n''$ or if $n' = n''$ and $\boldsymbol{k}'_* \neq \pm\boldsymbol{k}''_*$.

If the dimension $d > 1$ the dispersion relations may have symmetries additional to (24). If the dispersion relations $\omega_n(\boldsymbol{k})$ possess additional symmetries the admissible perturbations are supposed to have the symmetries too. Namely, if a finite group $G = \{\gamma_1, \ldots, \gamma_\kappa\}$ of transformations of the quasimomenta space $\mathbb{R}^d$ leaves the dispersion relation invariant, i.e. $\omega_n(\gamma\boldsymbol{k}) = \omega_n(\boldsymbol{k})$ for all γ from G, then the admissible perturbations must be invariant with respect to G, i.e. $\delta\omega_n(\gamma\boldsymbol{k}) = \delta\omega_n(\boldsymbol{k})$ for all γ from G. In this case the values of $\delta\omega_{n'}(\boldsymbol{k}')$ and $\delta\omega_{n''}(\boldsymbol{k}'')$ are independent in the vicinity of any two simple (that is not band-crossing) points $\boldsymbol{k}'_*$, $\boldsymbol{k}''_*$ if $n' \neq n''$ or if $n' = n''$ and $\boldsymbol{k}'_* \neq \gamma\boldsymbol{k}''_*$ for all possible γ from G.

Robust singularity of the interaction phase function. We consider the interaction phase $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (33) as a function of $(\boldsymbol{k}', \boldsymbol{k}'')$ that depends on the parameter $\boldsymbol{k}$. For a fixed $\vec{n}$ a singularity of $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ at a critical point $(\boldsymbol{k}'_*, \boldsymbol{k}''_*)$ satisfying (38), (39) with $\boldsymbol{k} = \boldsymbol{k}_*$, is called *robust* if it persists (as a solution of (38), (39) and as a singularity, may be with different $(\boldsymbol{k}_*, \boldsymbol{k}'_*, \boldsymbol{k}''_*)$) under all sufficiently small admissible perturbations of $\omega_n(\boldsymbol{k})$, $\omega_{n'}(\boldsymbol{k}')$, $\omega_{n''}(\boldsymbol{k}'')$, $\omega_{n'''}(\boldsymbol{k}''')$.

Generic interaction phase function. The interaction phase $\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (33) is said to *be generic* if it has only *robust* singularities at all its regular critical points $(\boldsymbol{k}'_*, \boldsymbol{k}''_*)$.

In other words, all possible degeneracies of $\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$, *which can be removed by an arbitrary small smooth, symmetry respecting perturbation of the dispersion relations, are assumed to be removed.*

Analysing patterns of singular behaviour of the interaction phase $\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ at critical points (41) we have singled out the following classes of important points $\vec{n}$ and $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''$, and, hence, corresponding quadruplets of interacting modes.

Diagonal quadruplets of modes (points). We have to consider separately the cases when some of the numbers n', n'' and n''' are equal and consequently some of the corresponding functions $\omega_{\bar{n}}(\boldsymbol{k})$, $\omega_{\bar{n}'}(\boldsymbol{k}')$, $\omega_{\bar{n}''}(\boldsymbol{k}'')$, $\omega_{\bar{n}'''}(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'')$ in (33) are not independent. Such cases may arise only when $n' = n''$ or $n' = n'''$, or $n'' = n'''$. We call a quadruplet of modes *positive diagonal* if at least one of the following relations is satisfied:

$$\{n' = n'',\ \boldsymbol{k}' = \boldsymbol{k}''\} \qquad \text{or} \qquad \{n' = n''',\ \boldsymbol{k}' = \boldsymbol{k}'''\} \qquad \text{or} \qquad \{n'' = n''',\ \boldsymbol{k}'' = \boldsymbol{k}'''\}. \tag{43}$$

Similarly, we call a quadruplet of modes *negative diagonal* if at least one of the following relations is satisfied:

$$\{n' = n'',\ \boldsymbol{k}' = -\boldsymbol{k}''\} \qquad \text{or} \qquad \{n' = n''', \boldsymbol{k}' = -\boldsymbol{k}'''\} \qquad \text{or} \qquad \{n'' = n''', \boldsymbol{k}'' = -\boldsymbol{k}'''\}. \tag{44}$$

If a quadruplet (point) is neither positive diagonal nor negative diagonal we will refer to it as a *non-diagonal* one.

Remark. Since the functions $\omega_{\bar{n}}(\boldsymbol{k})$ are 2π periodic, the equalities in (43), (44) and (36) are understood modulo $(2\pi\mathbb{Z})^d$; see the remark at the end of section 3.4 for more details.

Remark. If the dispersion relations have an additional symmetry, for example, they are invariant under the action of a group G, they can be treated similarly to the above. The analysis must take into account which points the functions $\omega_n(\boldsymbol{k})$ are independent. To this end the definition of a diagonal point should be modified in the following way. *A quadruplet is called diagonal with respect to a group* $G = \{\gamma_1, \ldots, \gamma_\kappa\}$ *of transformations* γ (G*-diagonal*), if there exists such γ in G that

$$\{n' = n'', \boldsymbol{k}' = \gamma\boldsymbol{k}''\} \qquad \text{or} \qquad \{n' = n''', \boldsymbol{k}' = \gamma\boldsymbol{k}'''\} \qquad \text{or} \qquad \{n'' = n''', \boldsymbol{k}'' = \gamma\boldsymbol{k}'''\}. \tag{45}$$

The group G is assumed to contain the group $G_0 = \{1, -1\}$ corresponding to the inversion symmetry (24). Evidently, (45) turns into (43), (44) if $G = G_0$. In this paper we consider in detail only the case $G = G_0$ and for larger groups G see section 3.4.

Our analysis of stronger cubic interactions singles out a special case of diagonal quadruplets of modes when for a given $\boldsymbol{k}$ a quadruplet of modes satisfies

$$n' = n'' = n''' = n. \tag{46}$$

In other words, the most strongly interacting quadruplets of modes satisfying the systems of selection rules (38), (39) belong to the same band. The quasimomenta $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}'''$ and the signs ζ in the index $\bar{n} = (\zeta, n)$ of $\omega_{\bar{n}}(\boldsymbol{k})$ for the strongly interacting modes satisfy (46) and one of the following systems of equations:

$$\{\boldsymbol{k}''' = -\boldsymbol{k}, \boldsymbol{k}' = \boldsymbol{k}'' = \boldsymbol{k}, \zeta_3 = -\zeta_0, \zeta_1 = \zeta_2 = \zeta_0\}, \tag{47}$$

Table 1. Indices of interaction at non-diagonal GVM–FM points of indicated type.

	Type of critical point		
Space dimension d	A_1	A_2	A_3
1	$q_0 = 1, d_c = 0$	None	None
2	$q_0 = 2, d_c = 1$	$q_0 = \frac{11}{6}, d_c = 0$	None
3	$q_0 = 3, d_c = 2$	$q_0 = \frac{17}{6}, d_c = 1$	$q_0 = \frac{11}{4}, d_c = 0$

Table 2. Indices of interaction at positive diagonal GVM–FM points of indicated type.

	Type of critical point				
Space dimension d	A_1	A_2	A_3	A_5	D_4
1	$q_0 = 1$ $d_c = 0$	None	None	None	None
2	$q_0 = 2$ $d_c = 1$	$q_0 = \frac{11}{6}$ $d_c = 0$	$q_0 = \frac{7}{4}$ $d_c = 0$	None	$q_0 = \frac{5}{3}$ $d_c = 0$
3	$q_0 = 3$ $d_c = 2$	$q_0 = \frac{17}{6}$ $d_c = 1$	$q_0 = \frac{11}{4}$ $d_c = 1$	$q_0 = \frac{8}{3}$ $d_c = 0$	$q_0 = \frac{8}{3}$ $d_c = 0$

or

$$\{\boldsymbol{k}' = -\boldsymbol{k}, \boldsymbol{k}'' = \boldsymbol{k}''' = \boldsymbol{k}, \zeta_2 = \zeta_3 = \zeta_0, \zeta_1 = -\zeta_0\}, \tag{48}$$

or

$$\{\boldsymbol{k}'' = -\boldsymbol{k}, \boldsymbol{k}' = \boldsymbol{k}''' = \boldsymbol{k}, \zeta_2 = -\zeta_0, \zeta_0 = \zeta_1 = \zeta_3\}. \tag{49}$$

Note, for instance, that if ζ are chosen as in (47) the interaction phase function (33) takes the form

$$\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \pm[-\omega_n(\boldsymbol{k}) + \omega_n(\boldsymbol{k}') + \omega_n(\boldsymbol{k}'') - \omega_n(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'')]. \tag{50}$$

Quadruplets of modes satisfying the relation (46) and one of the relations (47)–(49) are examples of what we call *double diagonal quadruplets of modes* (see the following section 2 for details).

The minimal values of the index $q_0 = q_0(\bar{n}, \bar{n}', \bar{n}'', \bar{n}''', \boldsymbol{k}'', \boldsymbol{k}', \boldsymbol{k})$ corresponding to the strongest interactions among all possible interacting quadruplets in a generic case occur exactly for the special, double diagonal quadruplets determined by (46), (47) or (46), (48) or (46), (49) (see tables 1–5 and corresponding sections of section 3 for details). In nonlinear optics this type of interactions is known to occur in the process of degenerate four-wave mixing, [12, p 232].

Remark. Note that for given $(\bar{n}, \boldsymbol{k})$ the selection rules (38) and (39) impose $2d + 1$ equations on $2d$ variables k', k'', and the fact that the system (38) and (39) (in contrast to the case of a quadratic nonlinearity; see [6]) always has a solution is due to the special structure of the function $\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined (50) or a similar representation for the cases (48) and (49).

Remark. Note that in (47) the $\{\boldsymbol{k}''' = -\boldsymbol{k}, \zeta''' = -\zeta\}$ values correspond to the eigenmode $(-\boldsymbol{k}, -\zeta)$ which is complex conjugate to $(\boldsymbol{k}, \zeta) = (\boldsymbol{k}', \zeta') = (\boldsymbol{k}'', \zeta'')$, and, since the relevant field is real valued, the corresponding mode amplitudes are also complex conjugate. This observation allows one to interpret the relations (46) and (47)–(49) as indicating that the self-interaction of a mode is among the strongest. It turns out that the described interaction is

Table 3. Indices of interaction at negative-diagonal GVM–FM points of indicated type.

	Type of critical point				
Space dimension d	A_1	A_2	A_3	A_4	D_4
1	$q_0 = 1$ $d_c = 0$	None	None	None	None
2	$q_0 = 2$ $d_c = 1$	$q_0 = \frac{11}{6}$ $d_c = 0$	$q_0 = \frac{7}{4}$ $d_c = 0$	None None	$q_0 = \frac{5}{3}$ $d_c = 1$
3	$q_0 = 3$ $d_c = 2$	$q_0 = \frac{17}{6}$ $d_c = 2$	$q_0 = \frac{11}{4}$ $d_c = 1$	$q_0 = \frac{27}{10}$ $d_c = 0$	$q_0 = \frac{8}{3}$ $d_c = 1$

Table 4. Indices of interaction at double positive diagonal GVM–FM points of indicated type.

	Type of critical point		
Space dimension d	A_1	D_4	$\tilde{Y}_5$
1	$q_0 = 1, d_c = 0$	None	None
2	$q_0 = 2, d_c = 1$	$q_0 = \frac{5}{3}, d_c = 0$	None
3	$q_0 = 3, d_c = 2$	$q_0 = \frac{8}{3}, d_c = 1$	$q_0 = \frac{5}{2}, d_c = 0$

Table 5. Indices of interaction of double negative or mixed diagonal GVM–FM points of a given type.

	Type of critical point				
Space dimension d	A_1	D_4	$T_{2,4,4}$	N_{16}	4-cubic
1	$q_0 = 1$ $d_c = 1$	$q_0 = \frac{2}{3}$ $d_c = 0$	None	None	None
2	$q_0 = 2$ $d_c = 2$	$q_0 = \frac{5}{3}$ $d_c = 1$	$q_0 = \frac{5}{2}$ $d_c = 0$	None	None
3	$q_0 = 3$ $d_c = 3$	$q_0 = \frac{8}{3}$ $d_c = 2$	$q_0 = \frac{5}{2}$ $d_c = 1$	$q_0 = \frac{12}{5}$ $d_c = 0$	$q_0 = \frac{7}{3}$ $d_c = 0$

intimately related to the nonlinear Schrödinger equation, a subject we intend to discuss in a subsequent paper.

It turns out the optical Kerr effect falls into the above framework. Indeed, the optical Kerr effect [12, p 26] involves a quadruplet of interacting modes with frequencies $-\omega_{\mathrm{S}}$, ω_{P}, $-\omega_{\mathrm{P}}$, ω_{S}. Let us consider a periodic medium with inversion symmetry satisfying

$$\omega_{\mathrm{S}} = \omega_n(\boldsymbol{k}) = \omega_{n'}(\boldsymbol{k}'), \qquad \omega_{\mathrm{P}} = \omega_{n'}(\boldsymbol{k}'') = \omega_{n'''}(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}''), \tag{51}$$

$$n''' = n'', \qquad n = n', \qquad \boldsymbol{k} = \boldsymbol{k}', \qquad \boldsymbol{k}''' = -\boldsymbol{k}'', \qquad \zeta_0 = \zeta_1 = \zeta_2 = -\zeta_3. \tag{52}$$

A direct examination shows that the relations (52) imply that the selection rule system (38) and (39) is satisfied.

We conclude this section by reminding that in this paper we consider only interactions satisfying the full system of selection rules (38) and (39), since analysis shows that all other interactions are weaker.

3. Classification of the critical points of the interaction phase function

In this section we provide a concise description of the results of a thorough asymptotic analysis of the cubic interaction integrals (32), (34) as $\varrho \to 0$ which serves as a basis for the classification of the strength of cubic nonlinear mode interactions. The asymptotic analysis, in turn, rests on the stationary phase method with the phase being the cubic interaction phase $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (33). In particular, we study here the properties of the selection rules system (38), (39) crucial to the asymptotic analysis.

3.1. Stationary phase method

The interaction integrals (32), (34) are oscillatory integrals with slowly varying amplitude A. To analyse their asymptotic behaviour as $\varrho \to 0$ we look at a more general oscillatory integral

$$I(\varrho, \boldsymbol{k}) = \int_{\mathbb{R}^{d_I}} \mathrm{e}^{\mathrm{i}\Phi(\boldsymbol{k},\boldsymbol{s})/\varrho} A(\boldsymbol{k}, \boldsymbol{s})\,\mathrm{d}\boldsymbol{s}, \qquad \boldsymbol{s} = (\boldsymbol{k}', \boldsymbol{k}''),\ \varrho \to 0, \tag{53}$$

where the integration variable $\boldsymbol{s}$ has dimension d_I, and $\boldsymbol{k}$ is a continuous parameter. Hence, generally speaking, we may view $\Phi(\boldsymbol{k}, \boldsymbol{s})$ as a d-parametric family of phase functions of $\boldsymbol{s}$. Obviously, (34) has the form of (53) where the integration vector-variable $\boldsymbol{s} = (\boldsymbol{k}', \boldsymbol{k}'')$ has the dimension $d_I = 2d$. The phase function $\Phi(\boldsymbol{k}, \boldsymbol{s})$ in the case of interest is the interaction phase function $\phi_{\vec{n}}$ defined by (33), i.e.

$$\Phi(\boldsymbol{k}, \boldsymbol{s}) = \phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''), \qquad \boldsymbol{s} = (\boldsymbol{k}', \boldsymbol{k}''), \tag{54}$$

where

$$\begin{aligned} &\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \zeta_0\omega_n(\boldsymbol{k}) - \zeta_1\omega_{n'}(\boldsymbol{k}') - \zeta_2\omega_{n''}(\boldsymbol{k}'') - \zeta_3\omega_{n'''}(\boldsymbol{k}'''), \\ &\boldsymbol{k}''' = \boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'', \qquad \zeta_i = \pm 1. \end{aligned} \tag{55}$$

Note that the integral (34) has the factor ϱ^{-1} which is not included in the integral (53).

Let us recall and review briefly the main concepts of the stationary phase method for oscillatory integrals of the form

$$I(\varrho) = \int_{\boldsymbol{R}^{d_I}} \mathrm{e}^{\mathrm{i}\Phi(\boldsymbol{s})/\varrho} A(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s}, \qquad \varrho \to 0, \tag{56}$$

where $A(\boldsymbol{s})$ is assumed to be an infinitely smooth function with a finite support. According to the stationary phase method, the main contribution to $I(\varrho)$ as $\varrho \to 0$ (up to ϱ^N with arbitrary large N) comes from small neighbourhoods of critical points of the phase $\Phi(\boldsymbol{s})$, that is the points $\boldsymbol{s}_*$ satisfying the equation

$$\nabla_{\boldsymbol{s}}\Phi(\boldsymbol{s}) = \boldsymbol{0}. \tag{57}$$

Observe that (57) implies the GVM rule (38) for $\Phi = \phi_{\vec{n}}$. Since (57) is a system of d_I equations for d_I variables, for a generic $\Phi(\boldsymbol{s})$ there is a finite number of such points. The integral over a small neighbourhood of a critical point $\boldsymbol{s}_*$ expands into an asymptotic series in powers of ϱ. The coefficients before the powers are written in terms of the values of $\Phi(\boldsymbol{s})$, $A(\boldsymbol{s})$ and their derivatives at the critical points $\boldsymbol{s}_*$. The most important is the matrix of the second-order derivatives, the so-called *Hessian*, defined by

$$\Phi''(\boldsymbol{s}_*) = \left\{ \frac{\partial^2\Phi(\boldsymbol{s}_*)}{\partial s_i\,\partial s_j} \right\}_{i,j=1}^{d_I}. \tag{58}$$

The simplest case is the so-called non-degenerate one when $\det\Phi''(\boldsymbol{s}_*) \neq 0$. In the non-degenerate case the integral (56) is of order $\varrho^{d_I/2}$ and its asymptotics are given by the classical formula; see the following section 'standard asymptotic approximations' (for more detail see [19]).

If the phase is degenerate and $\det \Phi''(s_*) = 0$ the integral (56) is significantly larger for small ϱ. More exactly, it is of order ϱ^q with $q > d_I/2 = d$.

The condition of being generic. We study here only 'generic' interaction phase functions $\Phi(s)$ that have a specific structure (54) with 'robust' degeneracies, with the terms 'generic' and 'robust' being explained in the previous section 2.2.1. In other words, we always assume that all possible degeneracies that can be removed by an arbitrary small admissible smooth perturbation of $\Phi(s)$ are removed and, consequently, we study only the critical points having robust singularities. For example, if $\Phi(s)$ is a generic phase function, then (54) has a finite number of solutions and $\det \Phi''(s_*) \neq 0$ at every critical point. Note though that one has to be cautious with generic functions depending on a parameter. For instance, when $\Phi(\boldsymbol{k}, s)$ is a generic function depending on a real vector (or scalar) parameter $\boldsymbol{k}$, then for every $\boldsymbol{k}$ equation (54) has a finite number of solutions, but one can no longer assert that $\det \Phi''(s_*) \neq 0$ for every $\boldsymbol{k}$! The analysis of oscillatory integrals involving external parameters strongly depends on the number of parameters. We consider here only cases when the external vector parameter $\boldsymbol{k}$ is one, two or three dimensional. The analysis crucially depends on the structure of functions we consider and their symmetries. Our main focus is on the interaction phase function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (54) and (55). *This interaction phase function even for generic $\omega_n(\boldsymbol{k})$ has a special structure, and cannot be considered as a generic phase function of* $(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$; *it can be considered as a generic function of all variables only at non-diagonal points.*

3.2. Standard asymptotic approximations and normal forms

In this section we collect some standard asymptotic formulae used in our analysis of the interaction integral (56).

One of the most important characteristics of a critical point s_* of the phase $\Phi(s)$ is the rank of the Hessian $\Phi''(s_*)$ defined by (58). We assume that that $\Phi(s)$ is analytic near s_*, $\nabla\Phi(s_*) = \mathbf{0}$, and $\Phi(s_*) = 0$. We also assume, without loss of generality, that the amplitude $A(s)$ is zero outside a small neighbourhood of the point s_*.

Non-degenerate points. A critical point s_* is called non-degenerate if the rank of the Hessian $\Phi''(s_*)$ is d_I. Choosing s_* as a new origin, we can reduce the phase function by a linear orthogonal change of variables to the canonical form

$$\Phi(s) = \mu_1 x_1^2 + \cdots + \mu_{d_I} x_{d_I}^2 + \mathrm{O}(|x|^3) \qquad \text{with } \mu_1 \neq 0, \ldots, \mu_{d_I} \neq 0, \tag{59}$$

where μ_j are the eigenvalues of the Hessian $\Phi''(s_*)$. Following the notations given by [3] we call such a point s_* a critical point of the type A_1. All the eigenvalues $\mu_1, \ldots, \mu_{d_I}$ of the Hessian $\Phi''(s_*)$ at this point are non-zero, and the following classical formula holds:

$$I(\varrho) = b_{A_1} \varrho^{\frac{d_I}{2}} A(s_*) + \mathrm{O}(\varrho^{\frac{d_I}{2}+1}), \qquad \varrho \to 0, \tag{60}$$

$$\begin{gathered} b_{A_1} = \frac{(2\pi)^{\frac{d_I}{2}}}{\sqrt{|\det \Phi''(s_*)|}} \exp\left\{\Phi(s_*) + \frac{\mathrm{i}\pi}{4}\mathrm{sign}[\Phi''(s_*)]\right\}, \\ \det \Phi''(s_*) = 2^{d_I} \mu_1 \cdots \mu_{d_I}, \ \mathrm{sign}\{\Phi''(s_*)\} = \sum_{j=1}^{d_I} \mathrm{sign}(\mu_j). \end{gathered} \tag{61}$$

Now we turn to more complicated types of *degenerate critical points* of the phase function when the rank of the Hessian $\Phi''(s_*)$ is less than d_I, and, equivalently, one or more of the eigenvalues $\mu_1, \ldots, \mu_{d_I}$ are zero. To find the leading term of the asymptotic expansion of the integral (56) in generic cases usually it is sufficient to know the leading polynomial of the Taylor expansion of the phase function. Since $\nabla\Phi(s_*) = \mathbf{0}$ the expansion starts with quadratic terms and when the quadratic part is degenerate the leading polynomial (the normal form) may include cubic and higher powers of variables x_j. We use the classification of degenerate critical points given in the theory of singularities and use the standard notation from this theory, in particular classes A_p, D_4 etc, as defined in [3].

Simplest degenerate points

Case 1. The rank of the Hessian is $d_I - 1$, that is exactly one of the eigenvalues of the Hessian vanishes, i.e. $\mu_{d_I} = 0$ and $\mu_1 \neq 0, \ldots, \mu_{d_I-1} \neq 0$. In this case we have a critical point s_* of the type A_p, with integer $p > 1$. That means that there exists an analytic change of variables $s = s_* + h(\xi)$ in the vicinity of the point $s = s_*$, $\xi = \mathbf{0}$ such that $\det h'(\mathbf{0}) = 1$ and $\Phi(s)$ can be written in terms of the variable $\xi = (\xi_1, \ldots, \xi_{d_I})$ as

$$\Phi(s) = \mu_1\xi_1^2 + \cdots + \mu_{d_I-1}\xi_{d_I-1}^2 + \beta\xi_{d_I}^{p+1}, \tag{62}$$

where the coefficient β equals $1/(p+1)!$ times $(p+1)$th derivative of $\Phi(s)$ along the direction of the null-space of $\Phi''(s_*)$. For a critical point of the class A_p the leading term of the asymptotic expansion takes the form (see, for example, [11, section 6.1])

$$I(\varrho) = \varrho^{\frac{d_I-1}{2}+\frac{1}{p+1}} b_{A_p} A(s_*) + \mathrm{O}\big(\varrho^{\frac{d_I-1}{2}+\frac{2}{p+1}}\big), \tag{63}$$

where the coefficient b_{A_p} is determined by the phase Φ at s_* by the following formula similar to (61):

$$b_{A_p} = \frac{2(2\pi)^{(n-1)/2}\Gamma(\frac{1}{p+1})}{(p+1)\beta^{\frac{1}{p+1}}\sqrt{|\det\Phi''_{d_I-1}(s_*)|}} \exp\left\{\frac{\mathrm{i}\pi}{4}\mathrm{sign}\{\Phi''_{d_I-1}(s_*)\} + \mathrm{sign}(\beta)\frac{\mathrm{i}\pi}{2p+2}\right\}, \tag{64}$$

where

$$\det\Phi''_{d_I-1}(s_*) = 2^{d_I-1}\prod_{j=1}^{d_I-1}\mu_j, \qquad \mathrm{sign}\{\Phi''_{d_I-1}(s_*)\} = \sum_{j=1}^{d_I-1}\mathrm{sign}(\mu_j). \tag{65}$$

Note that according to (63) the oscillatory index q_0 of the critical point of type A_p is

$$q_0 = q_{A_p} = \frac{d_I-1}{2} + \frac{1}{p+1}. \tag{66}$$

Formula (66) for the oscillatory index also holds for a non-degenerate point with $p = 1$.

Case 2. The rank of the Hessian is $d_I - 2$. In this case the simplest generic critical point s_* is of the type D_4. Namely, in a neighbourhood of the point s_* there exists an analytic change of variables $s = s_* + h(\xi)$ such that $\det h'(\mathbf{0}) = 1$ and $\Phi(s)$ can be written in terms of the variable $\xi = (\xi_1, \ldots, \xi_{d_I})$ as follows:

$$\Phi(s) = \mu_1\xi_1^2 + \cdots + \mu_{d_I-2}\xi_{d_I-2}^2 + \beta_1\xi_{d_I-1}^2\xi_{d_I} + \beta_2\xi_{d_I}^3 \tag{67}$$

where all $\mu_j \neq 0$, $j = 1, \ldots, d_I - 2$ and $\beta_1, \beta_2 \neq 0$. At a point of class D_4 the principal term of the asymptotic expansion of $I(\varrho)$ is given by

$$I(\varrho) = b_{D_4}\varrho^{q_0}A(s_*) + \mathrm{O}(\varrho^{q_0+\frac{1}{3}}), \qquad q_0 = \frac{(d_I-2)}{2} + \frac{2}{3} \tag{68}$$

with

$$b_{D_4} = \frac{(2\pi)^{(d_I-1)/2}}{\beta_1^{1/2}\beta_2^{1/6}\sqrt{|\det \Phi''_{d_I-2}(s_*)|}} \times \exp\left\{\frac{\mathrm{i}\pi}{4}\mathrm{sign}\{\Phi''_{d_I-2}(s_*)\}\right\}\frac{2\Gamma(\frac{1}{6})}{3}\cos\left\{\frac{\pi}{4}\mathrm{sign}(\beta_1\beta_2)+\frac{\pi}{12}\right\}. \tag{69}$$

When the rank of the Hessian is $d_I - 2$ and one of β_1, β_2 vanishes or the expansion of Φ with respect to ξ_{d_I-1}, ξ_{d_I} starts with the fourth or higher order terms, the singularities are even more complicated, but the expansions in generic cases look similar to (63), (68). We give the values of corresponding indices q_0 in the tables in section 2, omitting explicit expressions for the related coefficients b.

A general singularity can often be reduced to one of the standard cases based on the following Morse lemma which is fundamental in the study of critical points (see [3, 4]).

Morse lemma. *Let $\Phi(x_1, \ldots, x_n)$ be an analytic phase function in a neighbourhood of the origin. Assume that $\nabla\Phi(\mathbf{0}) = \mathbf{0}$, $\Phi(\mathbf{0}) = 0$ and the Hessian $\Phi''(\mathbf{0})$ has rank $n-k$ and is reduced by a linear change of variables to the form $\mu_1 x_1^2 + \cdots + \mu_{n-k} x_{n-k}^2$, with $\mu_1 \neq 0, \ldots, \mu_{n-k} \neq 0$. Then there exists an analytic change of variables $\boldsymbol{x} = h(\boldsymbol{\xi})$ in the neighbourhood of the origin with a unit linear part at zero such that*

$$\Phi(x_1, \ldots, x_n) = \mu_1\xi_1^2 + \cdots + \mu_{n-k}\xi_{n-k}^2 + f_1(\xi_{n-k+1}, \ldots, \xi_n) \tag{70}$$

with $\nabla f_1(\mathbf{0}) = \mathbf{0}$, $f_1''(\mathbf{0}) = \mathbf{0}$. The change of variables can always be taken so that

$$f_1(x_{n-k+1}, \ldots, x_n) = \Phi(0, \ldots, 0, x_{n-k+1}, \ldots, x_n). \tag{71}$$

For the asymptotic expansions of oscillatory integrals with a general phase function see [3, 4] and references therein. Our analysis shows that to find the leading terms of the asymptotic expansions of (40) in generic situations when $\boldsymbol{k}$ is a one-, two-, or three-dimensional variable it is sufficient to consider relatively simple singularities and the value of the index q_0 can be found in these cases.

3.3. Tables of interaction indices

In general, there are two types of critical points of the interaction phase function $\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (55): *regular critical points* and band-crossing points. A point $(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}''')$, $\boldsymbol{k}''' = \boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}''$, is called *band-crossing* in one of three cases:

(i) $\omega_{n'}(\boldsymbol{k}') = \omega_{n'+1}(\boldsymbol{k}')$ or $\omega_{n'}(\boldsymbol{k}') = \omega_{n'-1}(\boldsymbol{k}')$;
(ii) $\omega_{n''}(\boldsymbol{k}'') = \omega_{n''+1}(\boldsymbol{k}'')$ or $\omega_{n''}(\boldsymbol{k}'') = \omega_{n''-1}(\boldsymbol{k}'')$;
(iii) $\omega_{n'''}(\boldsymbol{k}''') = \omega_{n'''+1}(\boldsymbol{k}'')$ or $\omega_{n'''}(\boldsymbol{k}''') = \omega_{n'''-1}(\boldsymbol{k}''')$.

In this paper we do not consider band-crossing points, since their contribution to the interaction integrals is smaller than the contribution of regular ones. A regular critical point is not a band-crossing point, and at a regular critical point $\boldsymbol{k}'$, $\boldsymbol{k}''$, $\boldsymbol{k}'''$ all the functions $\omega_{\bar{n}'}(\boldsymbol{k}')$, $\omega_{\bar{n}''}(\boldsymbol{k}'')$, $\omega_{\bar{n}'''}(\boldsymbol{k}''')$ are smooth and the GVM condition (38) is satisfied. Triplets $(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ for which the FM rule (39) holds will be called *FM points*; regular critical points that satisfy (38) and (39) are called *GVM–FM points*. We remind that in this paper we only describe interactions of regular critical points which are also FM points. *In other words, we consider here only the points for which the entire selection rules system (38), (39) is satisfied.* For a discussion of band-crossing points, in particular for the extension of the GVM rule to band-crossing points, see [6, 7].

The analysis of the interaction phase function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ suggests one should consider separately the classes of non-diagonal and diagonal points.

The class of diagonal points, in turn, is partitioned into the following three subclasses.

Subclass of single diagonal points: such that among three pairs $(n', \boldsymbol{k}'; n'', \boldsymbol{k}'')$, $(n'', \boldsymbol{k}''; n''', \boldsymbol{k}''')$ and $(n', \boldsymbol{k}'; n''', \boldsymbol{k}''')$ there is exactly one pair which is diagonal (positive or negative).

Subclass of double diagonal points: such that among three pairs $(n', \boldsymbol{k}'; n'', \boldsymbol{k}'')$, $(n'', \boldsymbol{k}''; n''', \boldsymbol{k}''')$ and $(n', \boldsymbol{k}'; n''', \boldsymbol{k}''')$ there at least two pairs which are diagonal. Then, evidently, we have $n' = n'' = n'''$ and $\boldsymbol{k}'' = \pm\boldsymbol{k}'$, $\boldsymbol{k}''' = \pm(\mp)\boldsymbol{k}'$.

Subclass of zero diagonal points: $\boldsymbol{k}' = \boldsymbol{0}$, $\boldsymbol{k}'' = \boldsymbol{k} - \boldsymbol{k}'''$, or $\boldsymbol{k}'' = \boldsymbol{0}$, $\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{k}'''$, or $\boldsymbol{k}''' = \boldsymbol{0}$, $\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{k}''$.

The subclass of double diagonal points, in turn, can be partitioned into the following three subsubclasses.

Double positive diagonal points: $n' = n'' = n'''$ and $\boldsymbol{k}' = \boldsymbol{k}'' = \boldsymbol{k}''' = \boldsymbol{k}/3$.

Double negative diagonal points: $n' = n'' = n'''$ and $\boldsymbol{k}' = \boldsymbol{k}$, $\boldsymbol{k}'' = \boldsymbol{k}$, $\boldsymbol{k}''' = -\boldsymbol{k}$.

Mixed diagonal points: $n' = n'' = n'''$ and $\boldsymbol{k}' = \boldsymbol{k}$, $\boldsymbol{k}'' = -\boldsymbol{k}$, $\boldsymbol{k}''' = \boldsymbol{k}$ or $\boldsymbol{k}' = -\boldsymbol{k}$, $\boldsymbol{k}'' = \boldsymbol{k}$, $\boldsymbol{k}''' = \boldsymbol{k}$.

Remark. Note that we fix the numeration of the three pairs, and the above classification depends on the order of the three pairs $(n', \boldsymbol{k}'; n'', \boldsymbol{k}'')$, $(n'', \boldsymbol{k}''; n''', \boldsymbol{k}''')$ and $(n', \boldsymbol{k}'; n''', \boldsymbol{k}''')$. Change of variables $k''' \leftrightarrow k''$ switches the subsubclasses of the double negative diagonal and the mixed double diagonal points.

In what follows we give the final classification of the critical points if the interaction phase functions $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (55) are formed by generic dispersion relations $\omega_n(\boldsymbol{k})$. The details of their mathematical analysis are provided in section 3.

(1) *Non-diagonal points.* Possible types of critical points that may occur for generic dispersion relations and the values of the index q_0 in (37) are given in table 1.
Note that the set of points of a given type form a manifold of dimension d_c that depends on d. Points of classes A_1 arise when the Hessian of $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ with respect to $\boldsymbol{k}'$, $\boldsymbol{k}''$ is non-degenerate. Points of class A_2 and A_3 arise when the Hessian is degenerate and its null-space is one dimensional. The dimension d_c of the set of critical points of every type is given in the same table 1.

(2) *Single diagonal points.* Among positive diagonal points (to be concrete we take the diagonal pair $n'' = n'''$, $\boldsymbol{k}'' = \boldsymbol{k}'''$) generically FM critical points exist only when $\zeta'' = \zeta'''$. The index q_0 in (37) takes the values listed in table 2.
Among negative diagonal points (to be concrete we take the diagonal pair $n'' = n'''$, $\boldsymbol{k}'' = -\boldsymbol{k}'''$) generically FM critical points exist only when $\zeta'' = -\zeta'''$. The index q_0 in (37) takes the values listed in table 3.

(3) *Double positive diagonal.* In this case $n' = n'' = n'''$ and $\boldsymbol{k}' = \boldsymbol{k}'' = \boldsymbol{k}''' = \boldsymbol{k}/3$. If

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_0, \tag{72}$$

there can be GVM points satisfying the FM rule (39) which reduce to the following equation:

$$\omega_n(\boldsymbol{k}) = 3\omega_{n'}(\boldsymbol{k}/3). \tag{73}$$

Equation (73) can have solutions, and any such a solution would evidently correspond to the *third harmonic generation*. If equation (72) is not satisfied then generically there will be no FM critical points. The values of q_0 for the third harmonic generation and the dimension d_c of the corresponding set of critical points $(\boldsymbol{k}_*, \boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*)$ are provided in table 4.

Points of class D_4 exist when the Hessian $\omega''_{n''}(\frac{1}{3}\boldsymbol{k})$ has a one-dimensional null-space, points of class $\tilde{Y}_5$ exist when the third derivative in the direction of the null-space vanishes.

(4) *Double-negative diagonal.* In this case $n' = n'' = n'''$ and $\boldsymbol{k}' = \boldsymbol{k}, \boldsymbol{k}'' = \boldsymbol{k}, \boldsymbol{k}''' = -\boldsymbol{k}$ and the FM rule (39) takes the form

$$-\zeta_0\omega_n(\boldsymbol{k}) + (\zeta_1 + \zeta_2 + \zeta_3)\omega_{n'}(\boldsymbol{k}) = 0. \tag{74}$$

The strongest interactions occur when $n = n'$ (see section 4.2 for details). In the case $n = n'$, $\zeta_2 = \zeta_3$, generically there can be FM critical points if and only if $\zeta_2 = \zeta_0 = -\zeta_1$. The corresponding critical points are not degenerate and $q_0 = d$. In the case $\zeta_2 = -\zeta_3$, generically there can also be FM points, but one has to consider two subcases. First, if $\zeta_2 = -\zeta_1$, then (74) is satisfied if and only if $\zeta_3 = \zeta_0$, and the corresponding critical points are not degenerate and $q_0 = d$. Second, if $\zeta_0 = \zeta_1 = \zeta_2 = -\zeta_3$, then GVM and FM rules are satisfied for every $\boldsymbol{k}$ and we may have higher degeneration for some $\boldsymbol{k}$. The values of q_0 for this case are provided in table 5.

The first column in table 5 corresponds to $\boldsymbol{k}'$ values for which the Hessian $\omega''(\boldsymbol{k}')$ is non-degenerate; the second and the third correspond to $\boldsymbol{k}'$ values for which the Hessian $\omega''(\boldsymbol{k}')$ has a one-dimensional null-space; the fifth column corresponds to $\boldsymbol{k}'$ values for which the Hessian $\omega''(\boldsymbol{k}')$ has a two-dimensional null-space.

(5) *Double mixed diagonal.* In this case $n' = n'' = n'''$ and $\boldsymbol{k}' = \boldsymbol{k}, \boldsymbol{k}'' = -\boldsymbol{k}, \boldsymbol{k}''' = \boldsymbol{k}$. In the case $\zeta_2 = \zeta_3$, and if $n \neq n'$ then generically the FM rule cannot be satisfied. If $n = n'$, (74) is equivalent to $\zeta_0 = \zeta_2 = \zeta_3 = -\zeta_1$. In this case ϕ'' is generically non-degenerate and $q_0 = d$.

In the case $\zeta_2 = -\zeta_3$, there can be FM-GVM points, and one has to consider two subcases. In the first subcase, $\zeta_2 = -\zeta_1$, then (74) is satisfied if and only if $n = n'$ and $\zeta_3 = \zeta_0$, that is $\zeta_0 = \zeta_3 = \zeta_1 = -\zeta_2$. In this case ϕ'' can be degenerate, i.e. $\det \omega''_{n'}(\boldsymbol{k}) = 0$. In the second subcase, $\zeta_2 = \zeta_3$, that is $\zeta_0 = \zeta_1 = \zeta_2 = -\zeta_3$, generically if $n \neq n'$ the FM rule when generically cannot be satisfied. If under the same condition $n = n'$ then (74) is equivalent to $\zeta_0 = \zeta_2 = \zeta_3 = -\zeta_1$. In this case ϕ'' is generically non-degenerate and $q_0 = d$. The values of the indices of different points and the dimension of manifolds they form are presented in table 5. For details see section 4.3.

3.4. Additional symmetries

The obtained results take into account only the inversion symmetry (24). If the dispersion relations possess additional symmetries, *all the studied interactions are still strong and have the same values of the index* q_0. But the presence of additional symmetries may cause more interactions to become strong. Let $G = \{\gamma_1, \ldots, \gamma_\kappa\}$ be a finite group of those transformations

γ of the quasimomenta space $\mathbb{R}^d$ that leave all the dispersion relations $\omega_n(\boldsymbol{k})$ invariant, i.e. $\omega_n(\gamma \boldsymbol{k}) = \omega_n(\boldsymbol{k})$ for all n and $\boldsymbol{k}$. We assume that G includes the group G_0 corresponding to the inversion symmetry (24). For example, suppose that the medium is symmetric under reflection with respect to the plane $x_1 = 0$. Then in addition to (24) the dispersion relations will satisfy

$$\omega_n(k_1, k_2, k_3) = \omega_n(-k_1, k_2, k_3) \qquad \text{for all } n = 1, 2, \ldots \text{ and } \boldsymbol{k} \in [-\pi, \pi]^3. \tag{75}$$

Formula (75) means that $\omega_n(\boldsymbol{k})$ is invariant with respect to the transformation

$$R_1(k_1, k_2, k_3) = (-k_1, k_2, k_3). \tag{76}$$

In this case the group G consists of four elements

$$\gamma_1 = 1, \qquad \gamma_2 = -1, \qquad \gamma_3 = R_1, \qquad \gamma_4 = -R_1. \tag{77}$$

If G is a group larger than the group G_0 the set of 'diagonal points' defined by (45) consequently will be larger. The case of non-diagonal points is completely similar to the case of non-diagonal points considered in section 4.1 and table 1 remains the same. Now *single G-diagonal points* are such points that among three pairs $(n', \boldsymbol{k}'; n'', \boldsymbol{k}'')$, $(n'', \boldsymbol{k}''; n''', \boldsymbol{k}''')$ and $(n', \boldsymbol{k}'; n''', \boldsymbol{k}''')$ there is exactly one pair which is diagonal in the sense of (45). If the group G contains κ elements we have to consider κ different types of single diagonal points for different elements γ of the group G compared with two types (positive and negative) for the group G_0. *Double G-diagonal points* are such that among three pairs $(n', \boldsymbol{k}'; n'', \boldsymbol{k}'')$, $(n'', \boldsymbol{k}''; n''', \boldsymbol{k}''')$ and $(n', \boldsymbol{k}'; n''', \boldsymbol{k}''')$ there are at least two pairs which are diagonal. Then, evidently, we have $n' = n'' = n'''$ and $\boldsymbol{k}'' = \gamma_1 \boldsymbol{k}'$, $\boldsymbol{k}''' = \gamma_2 \boldsymbol{k}'$. Clearly, the third pair is also G-diagonal, $\boldsymbol{k}''' = \gamma_2 \gamma_1^{-1} \boldsymbol{k}''$. Now we have κ^2 different double diagonal cases (some of them are equivalent).

Zero G-diagonal points are the points where $\boldsymbol{k}' = \gamma \boldsymbol{k}'$ or $\boldsymbol{k}'' = \gamma \boldsymbol{k}''$ or $\boldsymbol{k}''' = \gamma \boldsymbol{k}'''$ for some γ in G.

The analysis of every diagonal, double diagonal and zero-diagonal case can be performed along the lines provided in the following section. Though additional symmetries can produce additional stronger interactions, we do not expect that there will be new types of singularities not covered in the tables given in the previous section.

Remark. The functions $\omega_n(k_1, k_2, k_3)$ are 2π periodic with respect to k_1, k_2 and k_3. We everywhere understand equalities $\boldsymbol{k}' = \boldsymbol{k}''$ etc modulo $(2\pi\mathbb{Z})^3$, that is components of $\boldsymbol{k}'$ and $\boldsymbol{k}''$ may differ by a vector $2\pi(n_1, n_2, n_3)$ where n_1, n_2, n_3 are integers. An expression of the form $\frac{1}{3}\boldsymbol{k}$ that occurs in a number of cases, for example in (73), denotes 3^d different vectors obtained by shifts of k_i by $\frac{2\pi}{3}$. The equations for diagonal points (43), (44) we have considered in the previous section and the phase matching equation (36) have integer coefficients, therefore they are invariant modulo $(2\pi\mathbb{Z})^3$ and the periodicity does not lead to additional diagonal points. At the same time, when more general groups G are considered one has to take the periodicity into account.

4. Mathematical analysis of GVM–FM points of the interaction phase function

This section is devoted to a rigorous and detailed mathematical analysis of the cubic interaction phase $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (55), we describe canonical types of singularities arising at critical points and the corresponding values of the interaction indices q_0. In the following sections we explain the classification of GVM–FM points given in tables 1–5 and give additional information on the points. We consider $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (55) as a function of $\boldsymbol{k}', \boldsymbol{k}''$ that depends on the parameter $\boldsymbol{k}$ in a neighbourhood of a point $(\boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*)$, $\boldsymbol{k}'''_* = \boldsymbol{k} - \boldsymbol{k}'_* - \boldsymbol{k}''_*$.

The point $(\boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*)$ may be non-diagonal, diagonal or double diagonal. Every different 'diagonal' case (non-diagonal, positive or negative diagonal, three double diagonal cases) is considered in a separate section. In every section we consider all possible types of degenerate GVM–FM points that a generic interaction phase $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ may have in the cases $d = 1, 2, 3$. In the diagonal and double diagonal cases the degeneracies depend on the signs ζ_i, therefore every section contains several subcases.

The analysis in every subcase consists of the following steps.

(1) We determine the dimension of the interaction manifold of GVM–FM points (of the diagonal type case under consideration) depending on $d = 1, 2, 3$; this manifold consists of solutions $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''$ of the selection rules system (38), (39) with fixed $\vec{n}$. When the Hessian $\phi'' = \phi''_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}'_*, \boldsymbol{k}''_*)$ of $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ has non-zero determinant GVM–FM points are non-degenerate or in the terminology of [3] are of type A_1 and their contribution to the interaction integral (34) can be found by the classical formula (60), (61), in this case $q_0 = d$.

(2) If the dimension of the above interaction manifold is greater than zero, that is the manifold includes a curve of GVM–FM points, the determinant of the Hessian can robustly vanish and degenerate points may exist. Therefore, we compute the Hessian (it is a $2d \times 2d$ symmetric matrix with a special structure that stems from the structure of the function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''))$ and determine in every subcase the dimension of a submanifold of GVM–FM points on which it degenerates.

(3) We determine if the dimension of the null-space of the Hessian can be greater than 1. At the points where the dimension of the null-space of the Hessian is one, the degenerate points are of type A_p, $p > 1$, and their contribution to the interaction integral (34) can be found using formula (63), (64), in this case $q_0 = d - \frac{1}{2} + \frac{1}{p+1}$.

(4) If the dimension of the null-space of the Hessian $\phi''_{\vec{n}}$ is greater than 1 we apply the Morse lemma to find the restriction of the phase function to the null-space of the Hessian and determine possible types of the degenerate points. The contribution of the degenerate points to the interaction integral (34) is given by the formula

$$I(\varrho) = b\varrho^{q_0} A(\boldsymbol{s}_*) + \mathrm{O}(\varrho^{q_1}), \qquad q_1 > q_0, \tag{78}$$

similar to (63) but with different values of coefficient b and index q_0. In the simplest case the degenerate critical point has type D_4 and (78) takes the form (68).

4.1. Degeneration of the Hessian at non-diagonal points

In this section we consider non-diagonal points as defined in section 2. Note that the selection rules system (38), (39) imposes $2d + 1$ constraints on $3d$ variables $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''$, and consequently it determines a $(d - 1)$-dimensional manifold $\Xi(\vec{n})$, $d = 1, 2, 3$, of points $(\boldsymbol{k}_*, \boldsymbol{k}'_*, \boldsymbol{k}''_*)$. The intersection of this manifold with a given $2d$-hyperplane $\boldsymbol{k}_* = \boldsymbol{k}_0$ is empty for a generic $\boldsymbol{k}_0$, and for some $\boldsymbol{k}_0$ it is a point or several points. We refer to the manifold $\Xi(\vec{n})$ as a *strong interactions manifold*.

Let us consider points $(\boldsymbol{k}_*, \boldsymbol{k}'_*, \boldsymbol{k}''_*)$ on the strong interaction manifold $\Xi(\vec{n})$ corresponding to the interaction function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ defined by (55). We show that for generic dispersion relations $\omega_n(\boldsymbol{k})$ at non-diagonal points the null-space of the Hessian $\phi''_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \phi''$ is at most one dimensional, implying that such points are always of the class A_p, $p \geqslant 1$, described in section 3.2. The Hessian ϕ'' is a $2d \times 2d$ matrix of the second derivatives of $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$

with respect to $\boldsymbol{k}'$, $\boldsymbol{k}''$ which can be written as follows:

$$\phi'' = -\begin{pmatrix} \zeta_1\omega''_{n'}(\boldsymbol{k}'_*) + \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*) & \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*) \\ \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*) & \zeta_2\omega''_{n''}(\boldsymbol{k}''_*) + \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*) \end{pmatrix}, \qquad \boldsymbol{k}'''_* = \boldsymbol{k}_* - \boldsymbol{k}'_* - \boldsymbol{k}''_*. \tag{79}$$

To study methods of degeneration of the Hessian ϕ'' we look at its null vectors $(\boldsymbol{v}', \boldsymbol{v}'')$ satisfying the following linear system of equations:

$$\begin{aligned} &\zeta_1\omega''_{n'}(\boldsymbol{k}'_*)\boldsymbol{v}' + \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*)(\boldsymbol{v}' + \boldsymbol{v}'') = \boldsymbol{0}, \\ &\zeta_2\omega''_{n''}(\boldsymbol{k}''_*)\boldsymbol{v}'' + \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*)(\boldsymbol{v}' + \boldsymbol{v}'') = \boldsymbol{0}. \end{aligned} \tag{80}$$

In the linear system (80) the vector parameter $(\boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*)$ takes values on the $(d-1)$-dimensional manifold $\Xi(\vec{n})$, $d - 1 \leqslant 2$. It is known (see, for example, [2]) that a generic symmetric matrix depending on two scalar parameters does not have a two-dimensional null-space. Notice that our matrix ϕ'' defined by (79) is not generic, but it has a special structure based on generic submatrices $\omega''_{n'}(\boldsymbol{k}'_*)$, $\omega''_{n''}(\boldsymbol{k}''_*)$, $\omega''_{n'''}(\boldsymbol{k}'''_*)$. To show that the system (80) has at most a one-dimensional null-space we have to consider several cases.

Case 1. The matrices $\omega''_{n'''}(\boldsymbol{k}'''_*)$, $\omega''_{n'}(\boldsymbol{k}'_*)$ and $\omega''_{n''}(\boldsymbol{k}''_*)$ are invertible. Then, we rewrite (80) in the form

$$\begin{aligned} &\boldsymbol{v}' + \zeta_1\zeta_3\omega''_{n'}(\boldsymbol{k}'_*)^{-1}\omega''_{n'''}(\boldsymbol{k}'''_*)(\boldsymbol{v}' + \boldsymbol{v}'') = \boldsymbol{0}, \\ &\boldsymbol{v}'' + \zeta_2\zeta_3\omega''_{n''}(\boldsymbol{k}''_*)^{-1}\omega''_{n'''}(\boldsymbol{k}'''_*)(\boldsymbol{v}' + \boldsymbol{v}'') = \boldsymbol{0}, \end{aligned} \tag{81}$$

that after elementary algebraic manipulation yields

$$[\zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*)^{-1} + \zeta_1\omega''_{n'}(\boldsymbol{k}'_*)^{-1} + \zeta_2\omega''_{n''}(\boldsymbol{k}''_*)^{-1}]\omega''_{n'''}(\boldsymbol{k}'''_*)(\boldsymbol{v}' + \boldsymbol{v}'') = \boldsymbol{0}. \tag{82}$$

Hence, if the $2d$-vector $(\boldsymbol{v}', \boldsymbol{v}'')$ is in the null-space of ϕ'', then d-vector $\boldsymbol{w} = \omega''_{n'''}(\boldsymbol{k}''')(\boldsymbol{v}' + \boldsymbol{v}'')$ is in the null-space of the $d \times d$ symmetric matrix $H_1(\boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*)$, i.e.

$$\begin{aligned} &H_1(\boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*)\boldsymbol{w} = \boldsymbol{0}, \\ &H_1(\boldsymbol{k}'_*, \boldsymbol{k}''_*, \boldsymbol{k}'''_*) = \zeta_1\omega''_{n'}(\boldsymbol{k}'_*)^{-1} + \zeta_2\omega''_{n''}(\boldsymbol{k}''_*)^{-1} + \zeta_3\omega''_{n'''}(\boldsymbol{k}'''_*)^{-1}. \end{aligned} \tag{83}$$

In the case of non-diagonal points $(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}''')$ the matrix $H_1(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}''')$ from (83) is the sum of three *independent* matrices $\omega''_n(\boldsymbol{k})^{-1}$, $\omega''_{n''}(\boldsymbol{k}'')^{-1}$, $\omega''_{n'''}(\boldsymbol{k}''')^{-1}$. Consequently, we can consider H_1 as a generic symmetric matrix depending on no more than two parameters (since $d - 1 \leqslant 2$). Hence, generically the dimension of the null-space of H_1 cannot be greater than one.

Let us show now that the null-space of ϕ'' should also be one dimensional. Observe that if the null-space of ϕ'' is two dimensional, the null-space of H_1 can be one dimensional only when $\boldsymbol{v}'_1 + \boldsymbol{v}''_1 = \boldsymbol{v}'_2 + \boldsymbol{v}''_2$ for two linearly independent null-vectors

$$(\boldsymbol{v}'_1, \boldsymbol{v}''_1) \nparallel (\boldsymbol{v}'_2, \boldsymbol{v}''_2) \tag{84}$$

of ϕ''. In this case the existence of a double-zero eigenvalue in (80) implies two equations for two pairs of vectors $(\boldsymbol{v}'_1, \boldsymbol{v}''_1)$ and $(\boldsymbol{v}'_2, \boldsymbol{v}''_2)$ with

$$\boldsymbol{v}'_1 + \boldsymbol{v}''_1 = \boldsymbol{v}'_2 + \boldsymbol{v}''_2. \tag{85}$$

From these equations we obtain the equations

$$\zeta_1\omega''_{n'}(\boldsymbol{k}')(\boldsymbol{v}'_1 - \boldsymbol{v}'_2) = \boldsymbol{0}, \qquad \zeta_2\omega''_{n''}(\boldsymbol{k}'')(\boldsymbol{v}''_1 - \boldsymbol{v}''_2) = \boldsymbol{0}, \qquad \boldsymbol{v}'_1 - \boldsymbol{v}'_2 = \boldsymbol{v}''_2 - \boldsymbol{v}''_1. \tag{86}$$

Note that (84) and (85) imply that $\boldsymbol{v}'_1 - \boldsymbol{v}'_2 = \boldsymbol{v}''_2 - \boldsymbol{v}''_1 \neq 0$, therefore both $\omega''_{n'}(\boldsymbol{k}')$, $\omega''_{n''}(\boldsymbol{k}'')$ are not invertible, and this is excluded in case 1.

Case 2. Exactly one of $\omega''_{n'}(\boldsymbol{k}')$, $\omega''_{n''}(\boldsymbol{k}'')$, $\omega''_{n'''}(\boldsymbol{k}''')$ is not invertible. First, we suppose that $\omega''_{n'}(\boldsymbol{k}')\boldsymbol{v}_1 = \mathbf{0}$ and that $\omega''_{n'}(\boldsymbol{k}')$ possesses a one-dimensional null-space (the case of non-invertible $\omega''_{n''}(\boldsymbol{k}'')$ is similar). From the first equation of (80) we get

$$\boldsymbol{v}_1 \cdot \omega''_{n'''}(\boldsymbol{k}''')(\boldsymbol{v}' + \boldsymbol{v}'') = \mathbf{0}. \tag{87}$$

If we set now $\boldsymbol{w} = \omega''_{n'''}(\boldsymbol{k}''')(\boldsymbol{v}' + \boldsymbol{v}'')$ then in view of (87) $\boldsymbol{w}$ belongs to the space $(\boldsymbol{v}_1)^{\perp}$ orthogonal to $\boldsymbol{v}_1$. Let us introduce a symmetric matrix $(\omega''_{n'}(\boldsymbol{k}'))^{-1}_{\perp}$ which acts on $(\boldsymbol{v}_1)^{\perp}$ as the inverse to the restriction of $\omega''_{n'}(\boldsymbol{k}')$ to $(\boldsymbol{v}_1)^{\perp}$, and is zero on $\boldsymbol{v}_1$. Then the vector $\zeta_1(\omega''_{n'}(\boldsymbol{k}'))^{-1}_{\perp}\boldsymbol{w}$ is well defined, belongs to $(\boldsymbol{v}_1)^{\perp}$ and, in view of (80), we have

$$\begin{aligned} &H_{10}\boldsymbol{w} = \mathbf{0}, \qquad \boldsymbol{w} \cdot \boldsymbol{v}_1 = 0, \\ &H_{10} = \zeta_1(\omega''_{n'}(\boldsymbol{k}'))^{-1}_{\perp} + \zeta_2\omega''_{n''}(\boldsymbol{k}'')^{-1} + \zeta_3\omega''_{n'''}(\boldsymbol{k}''')^{-1}. \end{aligned} \tag{88}$$

Note that the matrix H_{10} is a generic matrix as a function of $\boldsymbol{k}''$, consequently it has at most a one-dimensional null-space. Then, as in the previous case 1, we conclude that ϕ has at most a one-dimensional null-space too. Second, we suppose that $\omega''_{n'''}(\boldsymbol{k}''')$ is degenerate, and $\omega''_{n'''}(\boldsymbol{k}''')\boldsymbol{v}_1 = \mathbf{0}$. Now we proceed similarly to the case 1, but here we have to look at the possibility that vectors $\omega''_{n'''}(\boldsymbol{k}''')(\boldsymbol{v}' + \boldsymbol{v}'')$ form a one-dimensional set when $\boldsymbol{v}' + \boldsymbol{v}''$ is two dimensional. The latter is possible only if there is a solution of (80) that satisfies

$$\begin{aligned} &\boldsymbol{v}'_1 + \boldsymbol{v}''_1 = \boldsymbol{v}_1 \qquad \text{and} \qquad \omega''_{n'}(\boldsymbol{k}'_*)\boldsymbol{v}'_1 = \mathbf{0}, \\ &\omega''_{n''}(\boldsymbol{k}''_*)\boldsymbol{v}''_1 = \mathbf{0}, \qquad \omega''_{n'''}(\boldsymbol{k}'''_*)(\boldsymbol{v}'_1 + \boldsymbol{v}''_1) = \mathbf{0}. \end{aligned} \tag{89}$$

Therefore three determinants should equal zero at the same point on the two-dimensional manifold, which is impossible in a generic situation. Therefore, in case 2, the null-space of ϕ'' is at most one dimensional.

Case 3. Two or more of $\det\omega''_{n'}(\boldsymbol{k}')$, $\det\omega''_{n''}(\boldsymbol{k}'')$, $\det\omega''_{n'''}(\boldsymbol{k}''')$ are zero. This case can be robust only when $d = 3$ and can hold for several points on the strong interaction manifold. As in the case 2 we get that either $\boldsymbol{v}_1 \cdot \omega''_{n'''}(\boldsymbol{k}''')(\boldsymbol{v}' + \boldsymbol{v}'') = \mathbf{0}$ or $\boldsymbol{v}_2 \cdot \omega''_{n'''}(\boldsymbol{k}''')(\boldsymbol{v}' + \boldsymbol{v}'')$ which cannot hold for generic $\omega''_{n'''}(\boldsymbol{k}''')$.

Conclusion. In the case of non-diagonal points the Hessian ϕ'' generically cannot have two zero eigenvalues. For the space dimension $d = 2$ the generic interaction function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ may have several critical points of type A_2 at critical points. When $d = 3$ a robust curve of critical points of type A_2 and several points of type A_3 may exist.

4.2. Diagonal points of the interaction phase function

4.2.1. Hessian degeneration at positive diagonal points. Let us consider the diagonal $\boldsymbol{k}'' = \boldsymbol{k}'''$ and introduce new variables $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^d$ as follows:

$$\begin{aligned} &\boldsymbol{k}'' = \boldsymbol{\xi} + \boldsymbol{\eta}, \qquad \boldsymbol{k}' = \boldsymbol{k} - 2\boldsymbol{\xi}, \qquad \boldsymbol{k}''' = \boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'' = \boldsymbol{\xi} - \boldsymbol{\eta}, \\ &\boldsymbol{\xi} = \tfrac{1}{2}(\boldsymbol{k} - \boldsymbol{k}'), \qquad \boldsymbol{\eta} = \tfrac{1}{2}(\boldsymbol{k}'' - \boldsymbol{k}''') = \tfrac{1}{2}(2\boldsymbol{k}'' + \boldsymbol{k}' - \boldsymbol{k}). \end{aligned} \tag{90}$$

We will study the GVM rule and the Hessian degeneration in the new coordinates. The diagonal $\boldsymbol{k}'' = \boldsymbol{k}'''$ takes the form $\boldsymbol{\eta} = \mathbf{0}$, and the interaction function can be written as

$$\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \zeta_0\omega_n(\boldsymbol{k}) - \zeta_1\omega_{n'}(\boldsymbol{k} - 2\boldsymbol{\xi}) - \zeta_2\omega_{n''}(\boldsymbol{\xi} + \boldsymbol{\eta}) - \zeta_3\omega_{n'''}(\boldsymbol{\xi} - \boldsymbol{\eta}). \tag{91}$$

Then the GVM rule takes the form

$$\begin{aligned} &\nabla_{\boldsymbol{\xi}}\phi = 2\zeta_1\omega'_{n'}(\boldsymbol{k} - 2\boldsymbol{\xi}) - \zeta_2\omega'_{n''}(\boldsymbol{\xi} + \boldsymbol{\eta}) - \zeta_3\omega'_{n'''}(\boldsymbol{\xi} - \boldsymbol{\eta}), \\ &\nabla_{\boldsymbol{\eta}}\phi = -\zeta_2\omega'_{n''}(\boldsymbol{\xi} + \boldsymbol{\eta}) + \zeta_3\omega'_{n'''}(\boldsymbol{\xi} - \boldsymbol{\eta}). \end{aligned} \tag{92}$$

The Hessian ϕ'' with respect to $\boldsymbol{\xi}, \boldsymbol{\eta}$ takes the form

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}-2\boldsymbol{\xi}) + \zeta_2\omega''_{n''}(\boldsymbol{\xi}+\boldsymbol{\eta}) & \zeta_2\omega''_{n''}(\boldsymbol{\xi}+\boldsymbol{\eta}) - \zeta_3\omega''_{n'''}(\boldsymbol{\xi}-\boldsymbol{\eta}) \\ +\zeta_3\omega''_{n'''}(\boldsymbol{\xi}-\boldsymbol{\eta}) & \\ \zeta_2\omega''_{n''}(\boldsymbol{\xi}+\boldsymbol{\eta}) - \zeta_3\omega''_{n'''}(\boldsymbol{\xi}-\boldsymbol{\eta}) & \zeta_2\omega''_{n''}(\boldsymbol{\xi}+\boldsymbol{\eta}) + \zeta_3\omega''_{n'''}(\boldsymbol{\xi}-\boldsymbol{\eta}) \end{pmatrix}. \tag{93}$$

For $n'' = n'''$ and $\boldsymbol{\eta} = \mathbf{0}$ we get

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}-2\boldsymbol{\xi}) + (\zeta_2+\zeta_3)\omega''_{n''}(\boldsymbol{\xi}) & (\zeta_2-\zeta_3)\omega''_{n''}(\boldsymbol{\xi}) \\ (\zeta_2-\zeta_3)\omega''_{n''}(\boldsymbol{\xi}) & (\zeta_2+\zeta_3)\omega''_{n''}(\boldsymbol{\xi}) \end{pmatrix}. \tag{94}$$

At the diagonal $\boldsymbol{\eta} = \mathbf{0}$ we derive from (92) the GVM rule in the form

$$(\zeta_2-\zeta_3)\omega'_{n''}(\boldsymbol{\xi}) = \mathbf{0}, \qquad -2\zeta_1\omega'_{n'}(\boldsymbol{k}-2\boldsymbol{\xi}) + (\zeta_2+\zeta_3)\omega'_{n''}(\boldsymbol{\xi}) = \mathbf{0}. \tag{95}$$

We have two subcases $\zeta_2 - \zeta_3 = 0$ and $\zeta_2 + \zeta_3 = 0$.

Case 1. $\zeta_2 - \zeta_3 = 0$. Then (94) and (95) take the form

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}-2\boldsymbol{\xi}) + 2\zeta_2\omega''_{n''}(\boldsymbol{\xi}) & \mathbf{0} \\ \mathbf{0} & 2\zeta_2\omega''_{n''}(\boldsymbol{\xi}) \end{pmatrix}, \tag{96}$$

$$-2\zeta_1\omega'_{n'}(\boldsymbol{k}-2\boldsymbol{\xi}) + 2\zeta_2\omega'_{n''}(\boldsymbol{\xi}) = \mathbf{0}. \tag{97}$$

Note that in this case the GVM rule implies d equations on $2d$ variables $\boldsymbol{k}, \boldsymbol{\xi}$, and one more equation is imposed by the FM rule. Hence, there will be a $(d-1)$-dimensional set of points satisfying the selection rules system. In view of (96) the determinant of the Hessian ϕ'' is zero if and only if

$$\det[4\zeta_1\omega''_{n'}(\boldsymbol{k}-2\boldsymbol{\xi}) + 2\zeta_2\omega''_{n''}(\boldsymbol{\xi})]\,\det[\omega''_{n''}(\boldsymbol{\xi})] = 0. \tag{98}$$

when $d \geqslant 2$ this equation together with the GVM and FM rules yields generically a $(d-2)$-parametric family of solutions corresponding to degenerate critical points. Now we determine if the null-space of ϕ'' can be two dimensional. Note that a zero eigenvalue of ϕ'' given by (96) corresponds to solutions (u, v) of a system $Mu + Nu = \mathbf{0}$, $Nv = \mathbf{0}$ with $M = 2\zeta_2\omega''_{n''}(\boldsymbol{k}-2\boldsymbol{\xi})$, $N = 2\zeta_2\omega''_{n''}(\boldsymbol{\xi})$. The two-dimensional null-space of the system can be formed in two ways: either as a two-dimensional null-space of N or $M+N$ or as a sum of one-dimensional null-spaces of N or $M+N$. The first alternative is not robust when $d \leqslant 3$. The second alternative can be realized when both $\det N = 0$ and $\det(M+N) = 0$. This case is robust in a two-parameter family, therefore it can be realized when $d = 3$. Therefore, for $d = 3$, several robust points of type D_4 may exist with $q_0 = \frac{8}{3}$ and the leading term of the asymptotic expansion given by (68).

If the null-space of the Hessian ϕ'' is one dimensional, we introduce the new variables $\boldsymbol{p}, \boldsymbol{m}$

$$\boldsymbol{k}'' + \boldsymbol{k}''' = p, \qquad \boldsymbol{k}'' - \boldsymbol{k}''' = m, \qquad \boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{p} \tag{99}$$

yielding

$$\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \zeta_0\omega_n(\boldsymbol{k}) - \zeta_1\omega_{n'}(\boldsymbol{k}-\boldsymbol{p}) - \zeta_2\omega_{n''}(\tfrac{1}{2}(\boldsymbol{p}+\boldsymbol{m})) - \zeta_2\omega_{n''}(\tfrac{1}{2}(\boldsymbol{p}-\boldsymbol{m})). \tag{100}$$

Observe that the interaction phase function is even in $\boldsymbol{m}$. The corresponding Hessian ϕ'' in $\boldsymbol{p}, \boldsymbol{m}$ variables at $\boldsymbol{m} = \mathbf{0}$ takes the form

$$\phi'' = -\begin{pmatrix} \zeta_1\omega''_{n'}(\boldsymbol{k}-\boldsymbol{p}) + \frac{1}{2}\zeta_2\omega''_{n''}(\frac{1}{2}\boldsymbol{p}) & 0 \\ 0 & \frac{1}{2}\zeta_2\omega''_{n''}(\frac{1}{2}\boldsymbol{p}) \end{pmatrix}. \tag{101}$$

If $\det(\omega''_{n''}(\frac{1}{2}\boldsymbol{p})) = 0$ there is a zero eigenvector of ϕ'' of the form $(0, \boldsymbol{m}')$ with $\omega''_{n''}(\frac{1}{2}\boldsymbol{p})\boldsymbol{m}' = 0$, that is the null-space of ϕ'' lies in the $\boldsymbol{m}$ space. According to (100) the restriction of the phase ϕ to the line along the null-space direction with $\boldsymbol{p}$ fixed is even. Hence, the corresponding critical points are of the type A_3 or A_5. Points of the type A_3 form a manifold of the dimension $d_c = d - 2$, $d = 2, 3$. If $d = 3$ several points may have type A_5. If $\det[4\zeta_1\omega''_{n'}(\boldsymbol{k} - 2\boldsymbol{\xi}) + 2\zeta_2\omega''_{n''}(\boldsymbol{\xi})] = 0$ the points in generic cases may have types A_2 or A_3. In this case points of the type A_2 form a manifold of dimension $d_c = d - 2$, $d = 2, 3$, and there can be several points of the type A_3 if $d = 3$. The related interaction indices for all considered cases are presented in table 2.

Case 2. $\zeta_2 + \zeta_3 = 0$. The GVM rule (95) then has the form

$$\omega'_{n''}(\boldsymbol{\xi}) = \mathbf{0}, \qquad \omega'_{n'}(\boldsymbol{k} - 2\boldsymbol{\xi}) = \mathbf{0}. \tag{102}$$

If $n'' \neq n'$ or $\boldsymbol{k} - 2\boldsymbol{\xi} \neq \boldsymbol{k}$ (that is the point is not double diagonal) the relations (102) yield $2d$ equations for $2d$ variables and generically have a finite set of solutions. The FM rule generically cannot be satisfied. Hence, such points do not yield stronger interactions.

4.2.2. Hessian degeneration at negative diagonal points. Consider now the negative diagonal $n'' = n'''$, $\boldsymbol{k}'' = -\boldsymbol{k}'''$. Note that on this diagonal $\boldsymbol{k}' = \boldsymbol{k}$. It is convenient to introduce new variables $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$ by (90). In the new variables the negative diagonal takes the form $\boldsymbol{\xi} = \mathbf{0}$, and the interaction function $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''))$ is given by (91). Consequently, the GVM rule takes the form (92), and the Hessian ϕ'' with respect to $\boldsymbol{\xi}, \boldsymbol{\eta}$ is given by (93). At the diagonal $\boldsymbol{\xi} = \mathbf{0}$, $n'' = n'''$ we get from (92) and $\omega'_{n''}(-\boldsymbol{\eta}) = -\omega'_{n''}(\boldsymbol{\eta})$, which follows from (24), that the GVM rule takes the form

$$\begin{aligned} &-2\zeta_1\omega'_{n'}(\boldsymbol{k}) + (\zeta_2 - \zeta_3)\omega'_{n''}(\boldsymbol{\eta}) = \mathbf{0}, \\ &(\zeta_2 + \zeta_3)\omega'_{n''}(\boldsymbol{\eta}) = \mathbf{0}. \end{aligned} \tag{103}$$

Using the equality $\omega''_{n'''}(-\boldsymbol{\eta}) = \omega''_{n'''}(\boldsymbol{\eta})$, which follows from (24) we obtain from (93) at the negative diagonal

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}) + (\zeta_2 + \zeta_3)\omega''_{n''}(\boldsymbol{\eta}) & (\zeta_2 - \zeta_3)\omega''_{n''}(\boldsymbol{\eta}) \\ (\zeta_2 - \zeta_3)\omega''_{n''}(\boldsymbol{\eta}) & (\zeta_2 + \zeta_3)\omega''_{n''}(\boldsymbol{\eta}) \end{pmatrix}. \tag{104}$$

We have to consider here two subcases $\zeta_2 - \zeta_3 = 0$ and $\zeta_2 + \zeta_3 = 0$.

Case 1. $\zeta_2 - \zeta_3 = 0$. Then (103) takes the form

$$-2\zeta_1\omega'_{n'}(\boldsymbol{k}) = \mathbf{0}, \qquad 2\zeta_2\omega'_{n''}(\boldsymbol{\eta}) = \mathbf{0}. \tag{105}$$

If $n'' \neq n'$ then (105) yields $2d$ equations for $2d$ variables and has a finite set of solutions. If $n'' = n'$, $\boldsymbol{k} \neq \boldsymbol{\eta}$ we still have $2d$ equations for $2d$ variables. In both cases the FM rule cannot be generically satisfied.

Case 2. $\zeta_2 + \zeta_3 = 0$. Then (103) takes the form

$$-2\zeta_1\omega'_{n'}(\boldsymbol{k}) + 2\zeta_2\omega'_{n''}(\boldsymbol{\eta}) = \mathbf{0}. \tag{106}$$

In this case GVM implies d equations on $2d$ variables $\boldsymbol{k}, \boldsymbol{\eta}$, and one more equation is imposed by the FM rule, which takes the form $-\zeta_0\omega_n(\boldsymbol{k}) + \zeta_1\omega_{n'}(\boldsymbol{k}) = 0$ and has solutions only if $\zeta_0 = \zeta_1$. If $n = n'$ the FM rule is satisfied for any $\boldsymbol{k}$ and when $n \neq n'$ the set of GVM–FM points has dimension $d_c = d - 1$. The Hessian (104) takes the form

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}) & 2\zeta_2\omega''_{n''}(\boldsymbol{\eta}) \\ 2\zeta_2\omega''_{n''}(\boldsymbol{\eta}) & 0 \end{pmatrix}. \tag{107}$$

The determinant of ϕ'' is the zero if and only if

$$\det(\omega''_{n''}(\boldsymbol{\eta})) = 0. \tag{108}$$

Since $\boldsymbol{\eta} \neq 0$ (the so-called double zero-diagonal case $\boldsymbol{\eta} = 0$ when $\boldsymbol{k}'' = \boldsymbol{k}''' = 0$ is considered separately) the dependence on $\boldsymbol{\eta}$ is generic and if $n \neq n'$ we have a $(d-2)$-dimensional set of points of the type A_2, and for $d = 3$ we may have several points of the type A_3. If $n = n'$ we may have a $(d-1)$-dimensional set of points of the type A_2, a $(d-2)$-dimensional set of points of the type A_3, and for $d = 3$ we may have several points of the type A_4.

To determine whether the dimension of the null-space of ϕ'' can be greater than one we note the following. A generic two-parameter family of symmetric matrices $\omega''_{n''}(\boldsymbol{\eta})$ cannot have two zero eigenvalues. Let $2d$ vector $\vec{u} = (u, v)$ be in the null-space of ϕ'', $\phi''\vec{u} = 0$. We denote $M = 4\zeta_1\omega''_{n'}(\boldsymbol{k})$, $N = \zeta_2\omega''_{n''}(\boldsymbol{\eta})$, the matrices M and N are self-adjoint. The equation $\phi''\vec{u} = 0$ is equivalent to $Mu + Nv = \mathbf{0}$, $Nu = \mathbf{0}$. Since $\det N = 0$ we have $Nv_0 = \mathbf{0}$. Then $u = \mathbf{0}$, $v = v_0$ gives one solution of the system. The criterion of solvability of the equation $Mv_0 + Nv = \mathbf{0}$ is the condition $Mv_0 \cdot v_0 = 0$. If it is satisfied we get a solution v_1 of the equation $Mv_0 + Nv_1 = \mathbf{0}$ and a second solution $u = v_0$, $v = v_1$ of the system. Thus to get two non-zero solutions of the matrices M, N we have to satisfy the following two equations:

$$Nv_0 = \mathbf{0}, \qquad Mv_0 \cdot v_0 = 0. \tag{109}$$

The first has a non-zero solution when $\det N = \mathbf{0}$ and the second condition $Mv_0 \cdot v_0 = 0$ is a linear equation on the entries of M. Therefore, when $n \neq n'$ and $d = 3$ we may robustly have a two-dimensional null-space of ϕ'' at several GVM–FM points, they are generically points of type D_4. If $n = n'$ and $d = 2$ we may also have several GVM–FM points of type D_4. If $n = n'$ and $d = 3$ then the set of points of type D_4 may be one dimensional. The indices are collected in table 3, where we give the values of indices q_0 and dimension d_{c} that correspond to $n = n'$.

4.3. Double diagonal points of the interaction phase function

Now we consider double diagonal points for which $n' = n'' = n'''$ and $\boldsymbol{k}'' = \pm\boldsymbol{k}'''$, $\boldsymbol{k}''' = \pm(\mp)\boldsymbol{k}'$; different signs give rise to four cases. Since $\phi_{\bar{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ is invariant under exchange of $\bar{n}', \boldsymbol{k}'$ and $\bar{n}'', \boldsymbol{k}''$ it is sufficient to consider only one of two possible mixed cases, therefore we consider three combinations of signs: double positive, double negative and negative–positive.

Note that the integral (34) with $n' = n'' = n'''$ is preserved when $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are exchanged, as well as when $\boldsymbol{k}'$ and $\boldsymbol{k}'''$ are exchanged or $\boldsymbol{k}''$ and $\boldsymbol{k}'''$ are exchanged. Therefore to classify all the cases it is sufficient to consider the above pairs $(\boldsymbol{k}'', \boldsymbol{k}''')$ and $(\boldsymbol{k}', \boldsymbol{k}''')$; the pairs $(\boldsymbol{k}'', \boldsymbol{k}''')$ and $(\boldsymbol{k}', \boldsymbol{k}'')$, as well as $(\boldsymbol{k}', \boldsymbol{k}'')$ and $(\boldsymbol{k}', \boldsymbol{k}''')$ are similar.

4.3.1. Double positive diagonal. We consider here points $(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}''')$ on the intersection of the diagonals $\boldsymbol{k}'' = \boldsymbol{k}'''$ and $\boldsymbol{k}' = \boldsymbol{k}'''$. In this case $2\boldsymbol{k}'' = \boldsymbol{k} - \boldsymbol{k}'$, $3\boldsymbol{k}'' = \boldsymbol{k}$ and we again use the change of coordinates (90) where at the double positive diagonal point we get $\boldsymbol{\xi} = \frac{1}{2}(\boldsymbol{k} - \boldsymbol{k}') = \frac{1}{3}\boldsymbol{k}$, $\boldsymbol{\eta} = \mathbf{0}$.

Case 1. $\zeta_2 - \zeta_3 = 0$. The formulae (95) and (94) take respectively the form

$$(2\zeta_2 - 2\zeta_1)\omega'_{n'}(\tfrac{1}{3}\boldsymbol{k}) = \mathbf{0}, \tag{110}$$

$$\phi'' = -\begin{pmatrix} (4\zeta_1 + 2\zeta_2)\omega''_{n'}(\frac{1}{3}\boldsymbol{k}) & 0 \\ 0 & 2\zeta_2\omega''_{n'}(\frac{1}{3}\boldsymbol{k}) \end{pmatrix}. \tag{111}$$

Subcase 1.1. $\zeta_2 - \zeta_1 \neq 0$. Here we have several solutions $\boldsymbol{k}_*$ of (110) but the FM rule is not generically satisfied.

Subcase 1.2 (third harmonic generation). $\zeta_2 - \zeta_1 = 0$. In this case (110) holds for every $\boldsymbol{k}$ and ϕ'' is degenerate if and only if $\omega''_{n'}(\frac{1}{3}\boldsymbol{k})$ is. Here the FM rule takes the form

$$3\zeta_1\omega_{n'}(\tfrac{1}{3}\boldsymbol{k}) = \zeta_0\omega_n(\boldsymbol{k}), \tag{112}$$

and is satisfied only if $\zeta_1 = \zeta_0$. This case corresponds to the *third harmonic generation*.

The Hessian ϕ'' is block diagonal, so if $\det \omega''_{n'}(\boldsymbol{k}/3) = 0$ we have a two-dimensional null-space. The third-order Taylor polynomial $f_{(3)}$ of the interaction function restricted to the null-space can be written in appropriately chosen coordinates as in (121) (the analysis is similar to the case of the double negative diagonal and uses a modification of (122)) as

$$f_{(3)}(p_1, m_1) = \tfrac{1}{24}\omega'''_{n1}(\boldsymbol{k})[5p_1^3 + 3p_1m_1^2], \qquad \omega'''_{n1}(\boldsymbol{k}) = \partial^3\omega_n/\partial p_1^3. \tag{113}$$

The corresponding critical point belongs to the class D_4. Therefore, we have a point of class D_4 when $d = 2$ and a curve of points of class D_4 when $d = 3$. It is possible for $d = 3$ that $\omega'''_{n1}(\boldsymbol{k}) = 0$ at some point of the curve. In this case the principal part of the interaction function restricted to the null-space is given by a fourth-order polynomial of the form

$$\frac{2}{4!2^4}\omega''''_{n1}(\boldsymbol{k})(3p_1^2 + m_1^2)^2, \tag{114}$$

and the corresponding critical point belongs to the class $\tilde{Y}_5 = \tilde{T}_{2,5,5}$ with the normal form $(p^2 + m^2)^2 + ap^5$ (see [3, p 273]); the index $q_0 = \frac{5}{2}$ (see [4, p 185]).

Case 2. When $(\zeta_2 + \zeta_3) = 0$ (95) takes the form

$$\omega'_{n'}(\tfrac{1}{3}\boldsymbol{k}) = \boldsymbol{0}. \tag{115}$$

Equation (102) includes d equations for d variables and has a finite set of solutions. The FM rule cannot be generically satisfied. Such points do not contribute to stronger interactions.

Remark. As was mentioned in section 3.4, we identify vectors $\boldsymbol{k}', \boldsymbol{k}''$ etc modulo $(2\pi\mathbb{Z})^3$, therefore an expression of the form $\frac{1}{3}\boldsymbol{k}$ that occurs in (73), (110), (111), (115) denotes one of 3^d different vectors obtained by shifts of k_i by $\frac{2\pi}{3}$. For example, in (111) ϕ'' is considered separately at every point that represents $\frac{1}{3}\boldsymbol{k}$, a solution of (112) exists if one of the vectors $\frac{1}{3}\boldsymbol{k}$ solves it.

4.3.2. Double negative diagonal. We now consider points on the intersection of the negative diagonals $\boldsymbol{k}'' = -\boldsymbol{k}'''$ and $\boldsymbol{k}' = -\boldsymbol{k}'''$. In this case $\boldsymbol{k}'' = -\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}''$, $\boldsymbol{k}' = -\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}''$ that is $\boldsymbol{k} = \boldsymbol{k}' = \boldsymbol{k}'' = -\boldsymbol{k}'''$ and we again use the change of coordinates (90) where at the critical point we get $\boldsymbol{\xi} = \boldsymbol{0}$, $\boldsymbol{\eta} = \frac{1}{2}(2\boldsymbol{k}'' + \boldsymbol{k}' - \boldsymbol{k}) = \boldsymbol{k}$. The FM rule takes the form (74). The second differential (the Hessian) ϕ'' with respect to $\boldsymbol{\xi}, \boldsymbol{\eta}$ is given by (93) and for $\boldsymbol{\xi} = \boldsymbol{0}$ using that by (24) $\omega''_{n'}(-\boldsymbol{\eta}) = \omega''_{n'}(\boldsymbol{\eta})$ we obtain

$$\phi'' = -\begin{pmatrix} (4\zeta_1 + \zeta_2 + \zeta_3)\omega''_{n'}(\boldsymbol{k}) & (\zeta_2 - \zeta_3)\omega''_{n'}(\boldsymbol{k}) \\ (\zeta_2 - \zeta_3)\omega''_{n'}(\boldsymbol{k}) & (\zeta_2 + \zeta_3)\omega''_{n'}(\boldsymbol{k}) \end{pmatrix}. \tag{116}$$

At the diagonal $\boldsymbol{\xi} = \boldsymbol{0}$ we get from (92) using that by (24) $\omega'_{n'}(-\boldsymbol{\eta}) = -\omega'_{n'}(\boldsymbol{\eta})$

$$-2\zeta_1\omega'_{n'}(\boldsymbol{k}) + (\zeta_2 - \zeta_3)\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}, \qquad (\zeta_2 + \zeta_3)\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}. \tag{117}$$

We have two subcases $\zeta_2 - \zeta_3 = 0$ and $\zeta_2 + \zeta_3 = 0$.

Case 1. When $\zeta_2 - \zeta_3 = 0$ the GVM rule (103) takes the form

$$\omega'_{n'}(\boldsymbol{k}) = \mathbf{0}. \tag{118}$$

This system includes d equations for d variables and has a finite set of solutions; the additional FM rule (74) cannot be generically satisfied when $n \neq n'$. When $n = n'$, (74) is satisfied if and only if $\zeta_2 = \zeta_0 = -\zeta_1$. The critical point is not degenerate.

Case 2. When $\zeta_2 + \zeta_3 = 0$ (103) takes the form

$$(\zeta_2 - \zeta_1)\omega'_{n'}(\boldsymbol{k}) = \mathbf{0}. \tag{119}$$

When $\zeta_2 = -\zeta_1$ the GVM rule implies d equations on d variables $\boldsymbol{k}$, the FM equation (74) generically cannot be satisfied when $n \neq n'$. When $n = n'$, (74) is satisfied if and only if $\zeta_3 = \zeta_0$. In a generic case $\det(\omega''_{n''}(\boldsymbol{k})) \neq 0$, the Hessian ϕ'' is non-degenerate and we have a classical asymptotic given by (60) with $q_0 = d$.

When $\zeta_2 = \zeta_1$ the GVM rule is always satisfied, $\zeta_1 = \zeta_2 = -\zeta_3$ and (74) holds only when $\zeta_0 = \zeta_1 = \zeta_2 = -\zeta_3$. We have two different subcases $n \neq n'$ and $n = n'$. In the first subcase the FM rule determines a $(d-1)$-dimensional set of GVM–FM points and in the second all double diagonal points are GVM–FM points, that is the set is d dimensional; after taking this into account, the analysis is quite similar in both cases. Now we consider the case $n = n'$, obviously (47) holds and $\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')$ takes the form (50). According to (104) the Hessian is given by

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}) & 2\zeta_2\omega''_{n'}(\boldsymbol{k}) \\ 2\zeta_2\omega''_{n'}(\boldsymbol{k}) & 0 \end{pmatrix}. \tag{120}$$

To study the case when the Hessian ϕ'' is degenerate, that is $\det(\omega''_n(\boldsymbol{k})) = 0$, we introduce new variables

$$\boldsymbol{k}' + \boldsymbol{k}'' - 2\boldsymbol{k} = \boldsymbol{p}, \qquad \boldsymbol{k}' - \boldsymbol{k}'' = \boldsymbol{m}. \tag{121}$$

(Note that the determinant of this linear substitution equals 2^d, and this factor has to be taken into account if the oscillatory integral is evaluated in the $(\boldsymbol{p}, \boldsymbol{m})$ coordinates.) For simplicity we set $\zeta_0 = 1$ and using (24) we write

$$\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \omega_n(\boldsymbol{k}) - \omega_n(\boldsymbol{k} + \tfrac{1}{2}(\boldsymbol{p} + \boldsymbol{m})) - \omega_n(\boldsymbol{k} + \tfrac{1}{2}(\boldsymbol{p} - \boldsymbol{m})) + \omega_n(\boldsymbol{k} + \boldsymbol{p}). \tag{122}$$

This function is even with respect to $\boldsymbol{m}$. The axes of the $(\boldsymbol{p}, \boldsymbol{m})$ coordinate system are chosen in the following way. If the Hessian at $\boldsymbol{k} = \boldsymbol{k}_*$ has a zero eigenvalue, we take the axes p_1, m_1 in the direction of the null-space $\omega''_n(\boldsymbol{k})$ and axes p_2, m_2 (and p_3, m_3 for $d = 3$) in the direction of a vector corresponding to the non-zero eigenvalue λ_2 (respectively λ_3 for $d = 3$) of the Hessian $\omega''_{n'}(\boldsymbol{k}_*)$. For $d = 3$ we get at a GVM–FM point

$$-\phi_{\vec{n}}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \frac{\lambda_2}{4}[m_2^2 - p_2^2] + \frac{\lambda_3}{4}[m_3^2 - p_3^2] + f(\boldsymbol{p}, \boldsymbol{m}) \tag{123}$$

where $f(\boldsymbol{p}, \boldsymbol{m})$ has third-order zero at the origin and is even with respect to $\boldsymbol{m}$. According to the Morse lemma the class of the critical point is determined by the function $f(p_1, m_1)$ obtained by setting $p_2 = m_2 = p_3 = m_3 = 0$ in (122). The third-order Taylor polynomial $f_{(3)}$ of this function is of class D_4:

$$f_{(3)}(p_1, m_1) = \tfrac{1}{8}\omega'''_{n1}(\boldsymbol{k})[p_1 m_1^2 - p_1^3], \qquad \omega'''_{n1}(\boldsymbol{k}) = \partial^3\omega_n/\partial p_1^3. \tag{124}$$

If $\omega'''_{n1}(\boldsymbol{k}) \neq 0$, an invertible change of variables p_1, m_1 with a unit differential at zero reduces $f(p_1, m_1)$ to $f_{(3)}(p_1, m_1)$, we obtain the phase function of the type D_4, and the leading term of asymptotic expansion is given by formula (68). Note that the equation $\det \omega''_{n'}(\boldsymbol{k}) = 0$ determines a $(d-1)$-dimensional manifold of degenerate critical points. When $\omega'''_{n1}(\boldsymbol{k}) = 0$, the function $f(p_1, m_1)$ has fourth-order zero at the origin and can be reduced to its fourth-order Taylor polynomial

$$f_{(4)}(p_1, m_1) = \tfrac{1}{24}\omega''''_{n1}(\boldsymbol{k})\left[\tfrac{3}{4}p_1^2 m_1^2 + \tfrac{1}{8}m_1^4 - \tfrac{7}{8}p_1^4\right], \tag{125}$$

it belongs to the singularity class denoted by $T_{2,4,4}$ (see [3, p 260]). The oscillatory integral has asymptotic behaviour of order ϱ^{q_0-1} with $q_0 = \frac{5}{2}$ (see [4, p 185]). The manifold of such points, if non-empty, has dimension $d-2$ for $d = 2, 3$. If $d = 3$ the fourth-order coefficient $\omega''''_{n1}(\boldsymbol{k}) = \partial^4\omega_n/\partial p_1^4$ can vanish at several points and $f(p_1, m_1)$ has zero of the fifth order, its singularity type is then N_{16} (see [3, p 262]). The oscillatory integral has asymptotic behaviour of order ϱ^{q_0-1} with $q_0 = 2 + \frac{2}{5}$ (see theorems 6.4, 6.5 in [4]). The second possibility that may robustly occur in the three-dimensional case $d = 3$ is the existence of several points where the Hessian $\omega''_n(\boldsymbol{k})$ has a two-dimensional null-space. In this case the Morse lemma reduces the singularity of the phase to a function $f(p_1, m_1, p_2, m_2)$ of four variables that has a third-order zero at the origin (we call such a function 4-cubic in table 5) and in the generic case the oscillatory integral has asymptotic behaviour of order ϱ^{q_0-1} with $q_0 = 1 + \frac{4}{3}$ according to a theorem given by Varchenko (see theorem 6.4 in [4]).

4.3.3. Mixed double (positive and negative) diagonal. We consider now points $(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}''')$ on the intersection of the negative diagonal $\boldsymbol{k}'' = -\boldsymbol{k}'''$ and the diagonal $\boldsymbol{k}' = \boldsymbol{k}'''$. This case is similar to the case of a double-negative diagonal, therefore we discus it more briefly. In this case by (36) $\boldsymbol{k} = \boldsymbol{k}'$, $\boldsymbol{k}'' = -\boldsymbol{k}$ and we again use the change of coordinates (90) where at the double diagonal point we get $\boldsymbol{\xi} = \boldsymbol{0}$, $\boldsymbol{\eta} = \frac{1}{2}(2\boldsymbol{k}'' + \boldsymbol{k}' - \boldsymbol{k}) = -\boldsymbol{k}$. The FM rule, after taking into account (24), again takes the form (74).

Since by (24) $\omega'_{n'}(-\boldsymbol{\eta}) = -\omega'_{n'}(\boldsymbol{\eta})$, the GVM rule (92) takes the form

$$2\zeta_1\omega'_{n'}(\boldsymbol{k}) + (\zeta_2 - \zeta_3)\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}, \qquad (\zeta_2 + \zeta_3)\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}. \tag{126}$$

The Hessian ϕ'' with respect to $\boldsymbol{\xi}, \boldsymbol{\eta}$ is given by (93) and using the equality $\omega''_{n'}(-\boldsymbol{\eta}) = \omega''_{n'}(\boldsymbol{\eta})$, which follows from (24), we see that ϕ'' satisfies the relation (116).

We have two subcases $\zeta_2 - \zeta_3 = 0$ and $\zeta_2 + \zeta_3 = 0$.

Case 1. $\zeta_2 - \zeta_3 = 0$. The GVM rule (103) takes the form

$$\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}. \tag{127}$$

It yields d equations for d variables and has a finite set of solutions. If $n \neq n'$ the FM rule generically cannot be satisfied. If $n = n'$, then (74) is equivalent to $\zeta_0 = \zeta_2 = \zeta_3 = -\zeta_1$. In this case ϕ'' is generically non-degenerate and $q_0 = d$.

Case 2. $\zeta_2 + \zeta_3 = 0$. The relations (104) and (103) take, respectively, the form

$$\phi'' = -\begin{pmatrix} 4\zeta_1\omega''_{n'}(\boldsymbol{k}) & 2\zeta_2\omega''_{n'}(\boldsymbol{k}) \\ 2\zeta_2\omega''_{n'}(\boldsymbol{k}) & 0 \end{pmatrix}, \qquad (\zeta_1 + \zeta_2)\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}. \tag{128}$$

If $\zeta_1 = \zeta_2$ the GVM rule $\omega'_{n'}(\boldsymbol{k}) = \boldsymbol{0}$ gives d equations for d variables and has a finite set of solutions; the FM rule when $n \neq n'$ cannot be generically satisfied. When $n = n'$, (74) is equivalent to $\zeta_0 = \zeta_2 = \zeta_3 = -\zeta_1$. In the latter case ϕ'' is generically non-degenerate and $q_0 = d$.

If $\zeta_1 = -\zeta_2$ the GVM rule always holds. If $n \neq n'$, then FM rule determines a $d-1$ set of GVM–FM points. If $n = n'$, (74) is equivalent to $\zeta_0 = \zeta_3$, that is $\zeta_0 = \zeta_3 = \zeta_1 = -\zeta_2$. Clearly, in this case (49) holds. In this case ϕ'' can be degenerate if $\det\omega''_{n'}(\boldsymbol{k}) = 0$, the analysis of all possibilities is similar to the double negative diagonal case.

4.3.4. Zero-diagonal GVM–FM points. There is one more case when the symmetries restrict the generic variation of $\omega_{n'}(\boldsymbol{k}')$, namely near $\boldsymbol{k}' = \boldsymbol{0}$. Here we study the structure of the interaction phase function in the vicinity of such points. We consider here zero-diagonal points such that $\boldsymbol{k}' = -\boldsymbol{k}' = \boldsymbol{0}$. In this case $\boldsymbol{k}''' = \boldsymbol{k} - \boldsymbol{k}''$. At such a point thanks to (24)

$$\nabla\omega_{n'}(\boldsymbol{0}) = \boldsymbol{0} \tag{129}$$

and the GVM rule takes the form

$$\nabla\omega_{n'''}(\boldsymbol{k} - \boldsymbol{k}'') = \boldsymbol{0}, \qquad \nabla\omega_{n''}(\boldsymbol{k}'') = \boldsymbol{0}. \tag{130}$$

These two conditions together with the FM rule are not generically satisfied in a non-diagonal case. It can have a solution when $n''' = n''$ and one of the following conditions holds:

(i) $\boldsymbol{k} - \boldsymbol{k}'' = \boldsymbol{k}''$ (zero-positive diagonal) that is $\boldsymbol{k}'' = \boldsymbol{k}''' = \frac{1}{2}\boldsymbol{k}$;
(ii) $\boldsymbol{k} - \boldsymbol{k}'' = -\boldsymbol{k}''$ (zero-negative diagonal) that is $\boldsymbol{k} = \boldsymbol{k}' = \boldsymbol{0}$, $\boldsymbol{k}''' = -\boldsymbol{k}''$;
(iii) $\boldsymbol{k}'' = \boldsymbol{0}$ (zero-zero diagonal), $\boldsymbol{k} = \boldsymbol{k}'$.

In cases (i), (ii) the GVM rule takes the form

$$\nabla\omega_{n''}(\boldsymbol{k}'') = \boldsymbol{0} \tag{131}$$

yielding several solutions $\boldsymbol{k}''$ at the critical points of $\omega_{n''}(\boldsymbol{k}'')$.

(1) In the case of a zero-positive diagonal, the FM rule after taking into account (24), takes the form

$$\zeta_0\omega_n(\boldsymbol{k}) - \zeta_1\omega_{n'}(\boldsymbol{0}) - \zeta_2\omega_{n''}(\tfrac{1}{2}\boldsymbol{k}) - \zeta_3\omega_{n''}(\tfrac{1}{2}\boldsymbol{k}) = 0. \tag{132}$$

Since $\boldsymbol{k}$ takes several values determined by the GVM rule, the FM rule is not generically satisfied when $\boldsymbol{k} \neq 0$. For $\boldsymbol{k} = \boldsymbol{0}$ we get the zero-negative diagonal case that is considered in what follows.

(2) In the case of the zero-negative diagonal the FM rule

$$\zeta_0\omega_n(\boldsymbol{0}) - \zeta_1\omega_{n'}(\boldsymbol{0}) - \zeta_2\omega_{n''}(\boldsymbol{k}'') - \zeta_3\omega_{n''}(\boldsymbol{k}'') = 0 \tag{133}$$

is generically satisfied only when $n' = n$, $\zeta_0 = \zeta_1$, $\zeta_2 = -\zeta_3$. In particular, it is satisfied when $\boldsymbol{k} = \boldsymbol{k}' = \boldsymbol{k}'' = \boldsymbol{k}''' = \boldsymbol{0}$, in the latter case the sign variables ζ_i to enforce the FM rule should satisfy $\zeta_0 = \zeta_2$, $\zeta_1 = -\zeta_3$ or $\zeta_0 = \zeta_3$, $\zeta_1 = -\zeta_2$. The Hessian is generically non-degenerate and $q_0 = d$.

(3) In the case of the zero–zero diagonal the FM rule takes the form

$$\zeta_0\omega_n(\boldsymbol{k}) - \zeta_1\omega_{n'}(\boldsymbol{k}) - \zeta_2\omega_{n''}(0) - \zeta_3\omega_{n''}(0) = 0 \tag{134}$$

and is satisfied when $n = n'$, $\zeta_0 = \zeta_1$, $\zeta_2 = -\zeta_3$ for every $\boldsymbol{k}$. The GVM rule $\nabla\omega_{n'}(\boldsymbol{k}) = \boldsymbol{0}$ selects several $\boldsymbol{k}$. The Hessian generically is non-degenerate and $q_0 = d$.

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